Homework Solutions

48Displace $-|Q|$ from $z = 0$ to $z > 0$ with $z << R$;

$$F_{Q,z} = m\, a_z = -|Q|E_z = -|Q|\frac{kqz}{(R^2+z^2)^{\frac{3}{2}}} \approx -|Q|\frac{kqz}{R^3}$$

which is a harmonic oscillator equation, with

$$\omega^2 = \frac{kq|Q|}{mR^3} = \frac{|e|q}{4\pi\epsilon_0 mR^3}$$

49

$$|d\mathbf{F}_Q| = \frac{|kQ\,dq|}{R^2}\Big(-\sin\theta\mathbf{i} + \cos\theta\,\mathbf{j}\Big), \qquad dq = \frac{q}{\pi R}\,R\,d\theta = q\frac{d\theta}{\pi}$$

$$\mathbf{F}_Q = \int_0^\pi \frac{|kQq|}{\pi R^2}\Big(-\sin\theta\mathbf{i} + \cos\theta\,\mathbf{j}\Big)\,d\theta = -\frac{2|kQq|}{\pi R^2}\,\mathbf{i}$$

50In the absence of charges, the voltage everywhere is zero, or is at least everywhere some constant value (no fields mean no voltage **differences**), so

$$W_1 = q_1\Big(V(\mathbf{r}_1) - V(\infty)\Big) = q_1\Big(0-0\Big) = 0$$

Now the voltage at $\mathbf{r}_2$ is due to q_1 at $\mathbf{r}_1$;

$$W_2 = q_2\Big(V(\mathbf{r}_2) - V(\infty)\Big) = q_2\Big(\frac{kq_1}{|\mathbf{r}_2-\mathbf{r}_1|} - 0\Big) = \frac{kq_1q_2}{|\mathbf{r}_2-\mathbf{r}_1|}$$

Now the voltage at $\mathbf{r}_3$ is due to both q_1 and q_2;

$$V(\mathbf{r}_3) = \frac{kq_1}{|\mathbf{r}_3-\mathbf{r}_1|} + \frac{kq_2}{|\mathbf{r}_3-\mathbf{r}_2|}$$

so bring in q_3 from infinity;

$$W_3 = q_3\Big(V(\mathbf{r}_3) - V(\infty)\Big) = q_3\Big(\frac{kq_1}{|\mathbf{r}_3-\mathbf{r}_1|} + \frac{kq_2}{|\mathbf{r}_3-\mathbf{r}_2|}\Big)$$

and so proceding in this way, adding $W = W_1 + W_2 + W_3 + \cdots$ we see that we get a term

$$W_{ij} = \frac{kq_iq_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

for each distinct pair of charges in the system. For a square of charges there are six pairs (two diagonal, four sides) so

$$W = 4\frac{kq^2}{a} + 2\frac{kq^2}{\sqrt{2}\,a}$$

51

$$\mathbf{F}_Q = \frac{|kQq|}{\left(\sqrt{x^2+a^2}\right)^2}\Big(\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\Big) + \frac{|kQ(-q)|}{\left(\sqrt{x^2+a^2}\right)^2}\Big(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\Big)$$

$$= -2\frac{|kQq|}{\left(\sqrt{x^2+a^2}\right)^2}\Big(\sin\theta\,\mathbf{j}\Big) = -2\frac{|kQq|}{\left(\sqrt{x^2+a^2}\right)^2}\frac{a}{\sqrt{x^2+a^2}}\,\mathbf{j}$$

$$W = Q\Big(V(x) - V(0)\Big) = Q\Big(0-0\Big) = 0$$

since $V(x) = \frac{kq}{\sqrt{x^2+a^2}} + \frac{-kq}{\sqrt{x^2+a^2}}$ for any x.

52

The field (in the first octant) is a sum of three constant fields, each normal to one of the three planes

$$\mathbf{E} = \frac{\sigma_3}{2\epsilon_0}\mathbf{i} + \frac{\sigma_1}{2\epsilon_0}\mathbf{j} + \frac{\sigma_2}{2\epsilon_0}\mathbf{k} = 2\pi k\sigma_3\mathbf{i} + 2\pi k\sigma_1\mathbf{j} + 2\pi k\sigma_2\mathbf{k}$$

Make a path from $(0, 0, 0)$ to (a, b, c)

$$\mathbf{r}(t) = \Big(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\Big)\,t, \quad \mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}, \quad \mathbf{r}(1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$\frac{d}{dt}\mathbf{r}(t) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

and

$$W = -\int_0^1 Q\mathbf{E}\cdot\frac{d}{dt}\mathbf{r}(t)\,dt = -\int_0^1 Q\Big(2\pi k\sigma_3\mathbf{i} + 2\pi k\sigma_1\mathbf{j} + 2\pi k\sigma_2\mathbf{k}\Big)\cdot\Big(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\Big)\,dt$$

$$= -Q\int_0^1 \Big(2\pi k\sigma_3\,a + 2\pi k\sigma_1\,b + 2\pi k\sigma_2\,c\Big)\,dt = -Q\Big(2\pi k\sigma_3\,a + 2\pi k\sigma_1\,b + 2\pi k\sigma_2\,c\Big)$$

53

$$\mathbf{E} = \int_0^\infty \frac{k\,(\lambda\,dx)}{(x^2+a^2)^{\frac{2}{2}}}\Big(\frac{x\mathbf{i}}{\sqrt{x^2+a^2}} + \frac{a\mathbf{j}}{\sqrt{x^2+a^2}}\Big)$$

$$V = \int_0^\infty \frac{k\,(\lambda\,dx)}{(x^2+a^2)^{\frac{1}{2}}}$$

54

$$\sum F_x = 0 = -T\sin\theta + q\,2\pi k\sigma, \quad \sum F_y = 0 = T\cos\theta - mg$$

Divide

$$\tan\theta = \frac{2\pi kq\sigma}{mg}$$

55

The electric field there is

$$E_z = \frac{\sigma}{2\epsilon_0} = 2\pi k\sigma = -\frac{dV(z)}{dz}$$

$$\int_0^z dV(z) = -\int_0^z 2\pi k\sigma\,dz, \qquad V(z) - V(0) = -2\pi k\sigma\,z$$

$$V(z) = V(0) - 2\pi k\sigma\,z = V(0) - \frac{\sigma}{2\epsilon_0}\,z$$

56

The electric field within is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 R^3}\,\mathbf{r}$$

pick a path from the center to the surface, a radial ray (parallel to the field) of magnitude

$$|\mathbf{r}(t)| = R\,t, \qquad |\frac{d\mathbf{r}}{dt}| = R$$

Note that a point which is a distance r from the center is encountered at $t = \frac{r}{R}$ on this path.

$$V(r) - V(0) = -\int_0^{\frac{r}{R}} \mathbf{E}\cdot\frac{d\mathbf{r}(t)}{dt}\,dt = -\int_0^{\frac{r}{R}} |\mathbf{E}||\frac{d\mathbf{r}(t)}{dt}|\,dt = -\int_0^{\frac{r}{R}} \frac{q(Rt)}{4\pi\epsilon_0 R^3}\,R\,dt$$

$$= -\frac{1}{2}\frac{q}{4\pi\epsilon_0 R^3}\,R^2\left(\frac{r}{R}\right)^2 = -\frac{qr^2}{8\pi\epsilon_0 R^3}$$