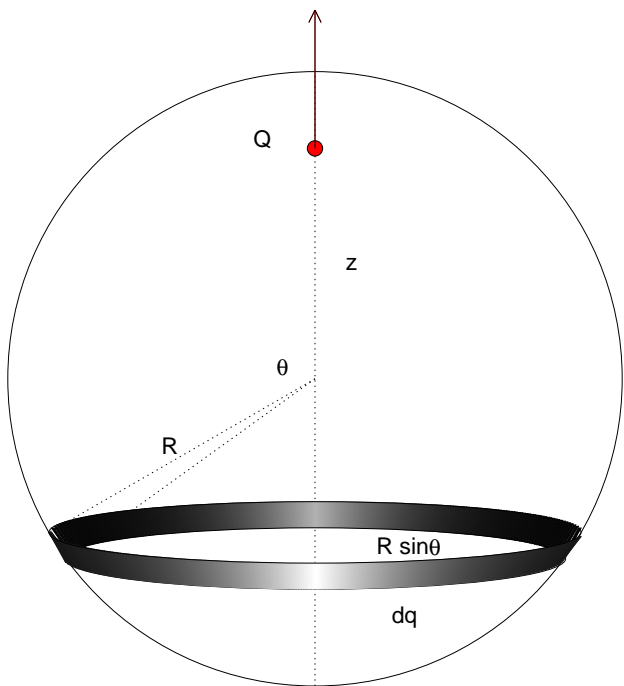


6 Gauss' Law

I will now present two important properties of highly symmetrical electric fields that will pave the way for important applications of Gauss' law.

Example 47



Compute the electrostatic potential energy of a charge Q inside of a thin-skinned hollow spherical shell of radius R and charge q .

Subdivide the shell into a stack of rings, one seen nearly edge-on in the figure. Each ring has radius $r = R \sin \theta$, is a distance $z + R \cos \theta$ from Q , and if cut and laid flat would be a strip of area $dA = (R d\theta)(2\pi R \sin \theta)$ and charge

$$dq = \rho dA = \frac{q}{4\pi R^2} 2\pi R^2 \sin \theta d\theta = \frac{q \sin \theta d\theta}{2}$$

The potential energy of Q in the field of the stack of rings is

$$U(z) = \int_0^\pi \frac{dq Q}{4\pi\epsilon_0 \sqrt{(z + R \cos \theta)^2 + R^2 \sin^2 \theta}}$$

$$= \int_0^\pi -\frac{Q}{4\pi\epsilon_0 \sqrt{(z + R \cos \theta)^2 + R^2 \sin^2 \theta}} \frac{q \sin \theta d\theta}{2} = \frac{qQ}{8\pi\epsilon_0 R z} (|R - z| - |R + z|)$$

If $z < R$, this results in

$$U = QV(z) = \frac{qQ}{8\pi\epsilon_0 R z} ((R - z) - (R + z)) = \frac{qQ}{4\pi\epsilon_0 R}$$

a result completely independent of z , **therefore inside of this hollow sphere there is no electric field and Q experiences no force**

$$F_z = -Q \frac{\partial V}{\partial z} = 0$$

If $z > R$ we obtain

$$U = QV(z) = \frac{qQ}{8\pi\epsilon_0 R z} ((z - R) - (R + z)) = \frac{qQ}{4\pi\epsilon_0 z}$$

and Q experiences a force

$$F_z = -Q \frac{\partial V}{\partial z} = \frac{qQ}{4\pi\epsilon_0 z^2}$$

which is exactly the same as if all of the charge q comprising the shell were concentrated at its center.

We saw an identical property for gravitational fields.

Example 48

Find the force exerted by a solid ball of charge of density ρ (in Coulombs per cubic meter) of total charge q , radius R on a point charge Q located **inside** of it a distance $r < R$ from the center.

The previous examples suggest dividing the ball into concentric shells of radii ξ , thickness $d\xi$ and charge $dq(\xi) = (4\pi\xi^2)(d\xi)\frac{q}{4\pi R^3} = 3q\frac{\xi^2 d\xi}{R^3}$, each acting like a point charge concentrated at the center of the sphere and exerting

$$dF_r = \frac{Q dq(\xi)}{4\pi\epsilon_0 r^2} \quad \text{if } \xi < r$$

on Q . Adding up the forces exerted by these shells we arrive at

$$F_r = \int_0^r \frac{Q}{4\pi\epsilon_0 r^2} 3 \frac{\xi^2 d\xi}{R^3} = \frac{qQ r}{4\pi\epsilon_0 R^3} = Q |\mathbf{E}|_{r,ball}$$

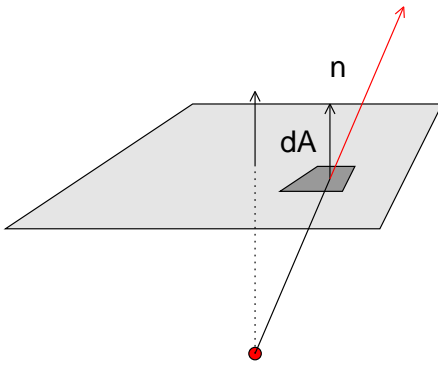
which vanishes at the center of the ball ($r = 0$). We find that within the ball of charge the electric field is

$$|\mathbf{E}(\mathbf{r})| = \frac{qr}{4\pi\epsilon_0 R^3}, \quad r = |\mathbf{r}| < R$$

and outside of the surface of the ball, the whole thing acts as though its charge is concentrated at the center

$$|\mathbf{E}(\mathbf{r})| = \frac{q}{4\pi\epsilon_0 r^2}, \quad r = |\mathbf{r}| > R$$

Gauss' law is an identity that under circumstances of extremely high symmetry, can be used to construct the strength of the electric field on an equipotential surface. In this usage it is somewhat limited in range of applicability. In addition one needs to know what the field geometry will be in advance. Nevertheless it is very useful in that it is a way of getting an answer without having to perform potentially nasty multiple integrals.



We define the **flux** of a field through an oriented surface to be the number of field lines that pass through the surface. The flux is positive if the field lines cross the surface in the same direction as the normal, and negative otherwise. Mathematically we say it this way; let dA be an area element with normal \mathbf{n} . Then the electric flux through the area element is

$$\mathcal{F} = \mathbf{E} \cdot \mathbf{n} dA$$

The statement of Gauss' law is that for a closed, orientable surface S ,

$$\oint_S \mathbf{E} \cdot \mathbf{n} dA = \frac{q_{inS}}{\epsilon_0}$$

where q_{inS} is all the charge enclosed by the surface S . It is simple to prove, but we will just offer two examples as "proof".

Example 49

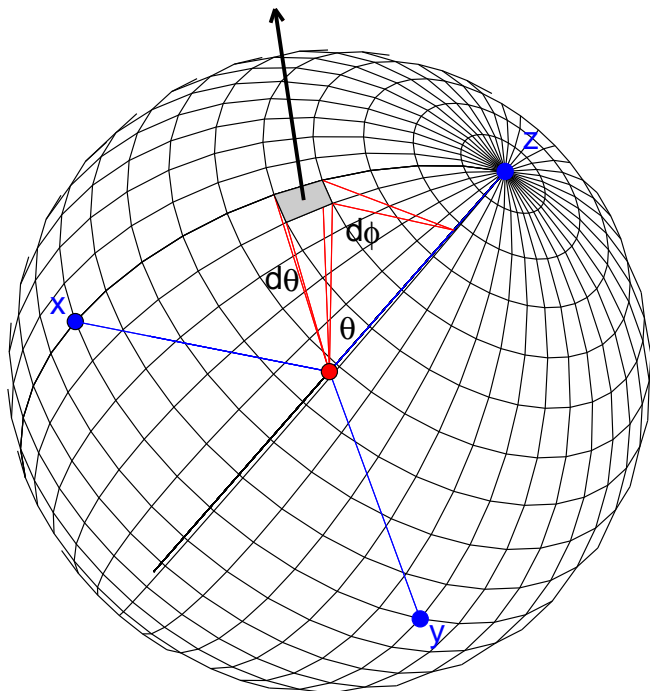
Surround a point charge at the origin with a sphere of radius r . The normal to this sphere is $\mathbf{n} = \mathbf{r}$ which is parallel to the field on the surface

$$\mathbf{E}(r) = \frac{q}{4\pi\epsilon_0 r^2} \mathbf{r}$$

and so the flux of the charge q 's field through the sphere is

$$\begin{aligned} \mathcal{F} &= \oint_S \mathbf{E} \cdot \mathbf{r} dA = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta r^2 \frac{q}{4\pi\epsilon_0 r^2} \mathbf{r} \cdot \mathbf{r} \\ &= \frac{q}{4\pi\epsilon_0} 4\pi = \frac{q}{\epsilon_0} \end{aligned}$$

which is precisely Gauss' law since S contains q .



Example 50

Compute the flux of a constant electric field $\mathbf{E} = E\mathbf{x} = E\mathbf{i}$ through the cube of side a whose sides have normals in all six cardinal directions $\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$.

The cube has 6 sides of areas a^2 with various normals. There are six contributions to the flux

$$\begin{aligned} \mathcal{F}_{top} &= (E\mathbf{i}) \cdot (a^2\mathbf{k}), & \mathcal{F}_{bottom} &= (E\mathbf{i}) \cdot (a^2(-\mathbf{k})) \\ \mathcal{F}_{front} &= (E\mathbf{i}) \cdot (a^2\mathbf{j}), & \mathcal{F}_{back} &= (E\mathbf{i}) \cdot (a^2(-\mathbf{j})) \\ \mathcal{F}_{left} &= (E\mathbf{i}) \cdot (a^2(-\mathbf{i})), & \mathcal{F}_{right} &= (E\mathbf{i}) \cdot (a^2\mathbf{i}) \end{aligned}$$

Four terms will be zero since the electric field is perpendicular to the normals to four faces of the cube. Two terms are nonzero but cancel one another since the flux through the left side is negative (enters S) and the flux through the right side is positive (leaves S). The result is $\mathcal{F} = 0$ because there is no net charge inside of S .

We think of positive charges as **sources** of field lines and negative charges as **sinks**. Gauss' law gives the net number of sources minus number of sinks within a surface, with each source having a value of $\frac{q}{\epsilon_0}$. Another way to look at it is that each source produces $\frac{q}{\epsilon_0}$ field lines and Gauss' law counts up how many **leave** a closed surface. **We can think of q as a source of $\frac{q}{\epsilon_0}$ field lines.**

We can use Gauss' law to compute the field strength **on an equipotential** if the strength is everywhere the same on the equipotential. This is only true in cases of very high symmetry, such as encountered with point, spherical, linear or planar charge distributions. In other words the technique is very restricted in applicability, but it can always be used to estimate field strengths.

In order to apply Gauss' law to this end, we choose S (called a Gaussian surface) to be an equipotential for the charge distribution at hand. It is true that equipotential surfaces always have the

symmetry of the charge distribution. Spherical distributions have spherical equipotentials, linear distributions have cylindrical equipotentials.

On any equipotential the electric field is normal and so on S it will be true that

$$\mathbf{E} = |\mathbf{E}| \mathbf{n}$$

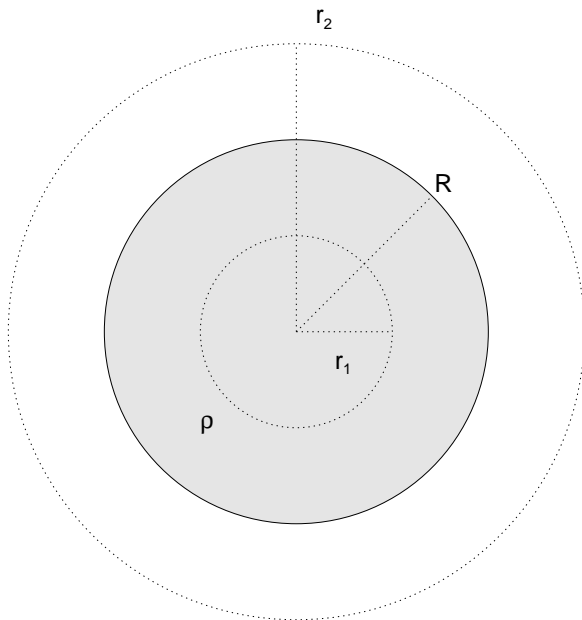
If the system has sufficient symmetry, the electric field strength will be constant all over S , and so can be extracted from the flux integral

$$\mathcal{F} = \oint_S |\mathbf{E}| \mathbf{n} \cdot \mathbf{n} dA = |\mathbf{E}| \oint_S dA = |\mathbf{E}| A_S$$

where A_S is the area of the Gaussian surface. We have that

$$|\mathbf{E}| A_S = \frac{q_{inS}}{\epsilon_0}$$

provides us with the field strength on S .



Example 51

A solid ball of radius R is made of charged dust of charge density ρ in $\frac{\text{coul}}{\text{m}^3}$. Compute the electric field strength a distance $r < R$ from its center.

Equipotentials are spheres (dotted), so construct a concentric Gaussian surface S within the distribution. It contains a total charge

$$q_{inS} = \rho \frac{4\pi r^3}{3} \quad \text{and has area} \quad A_S = 4\pi r^2$$

and so we find

$$|\mathbf{E}(r_1)| = \frac{q_{inS}}{A_S \epsilon_0} = \frac{\rho r_1}{3\epsilon_0}$$

meaning that there is no field at the center of the sphere, a result obvious from symmetry considerations alone.

For $r > R$ a Gaussian equipotential will contain the entire charge

$$q_{inS} = \rho \frac{4\pi R^3}{3} = Q_{total}$$

of the ball, and again have an area $A_s = 4\pi r^2$ and we find for $r > R$

$$|\mathbf{E}(r_2)| = \frac{q_{inS}}{A_s \epsilon_0} = \frac{Q_{total}}{4\pi \epsilon_0 r_2^2}$$

which looks like the field of a point charge at the origin. We have recovered a useful fact that we originally learned in the context of Newtonian gravitational fields; The field **external** to a spherical

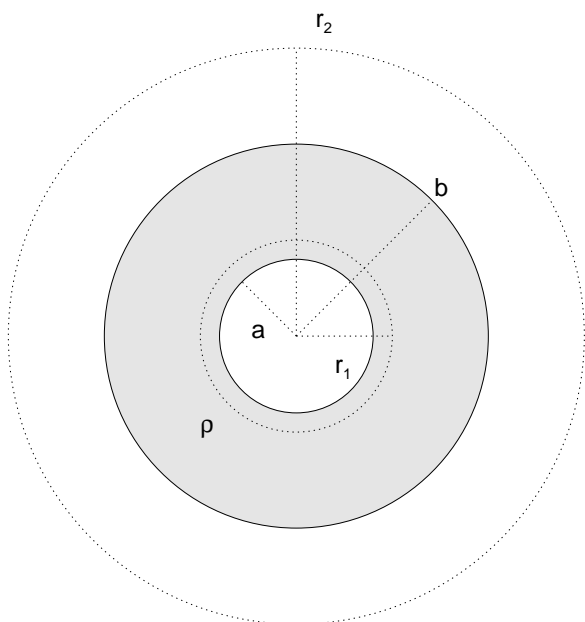
charge distribution is the same as if all of the charge were concentrated at the origin.

Example 52

A hollow ball of inner radius a and outer radius b is made of charged dust of density ρ . Compute the electric field strength for $r < a$. Equipotentials will be spheres, so construct a concentric sphere of radius $r < a$. Notice that it contains **no** charge, Therefore

$$|\mathbf{E}(r)| = 0, \quad r < a$$

a result that could be verified by integration but only with great difficulty. Since there is no electric field within this cavity, the voltage will be constant inside of the ball. **We have already seen this in a previous but much more difficult calculation.**



Example 53

Use Gauss' law to compute the field strength everywhere for an infinite sheet of charge of surface density σ

Equipotentials are planes parallel to the sheet. We construct a Gaussian surface S made of two areas A on equipotentials and close it off by adding a tubular area whose walls have a normal perpendicular to the electric field. Since nothing fluxes through these walls they do not contribute to the integration in the Gauss' law expression.

The end caps of area A each have **outward** directed normals in the same direction as the electric field, and so there will be a total flux of

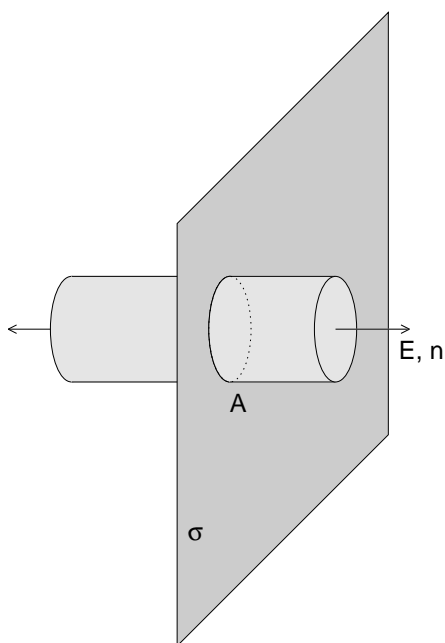
$$\mathcal{F} = 2A|\mathbf{E}|$$

The total charge within S all lies on the intersection of S with the plane; an area A with charge per area σ and so

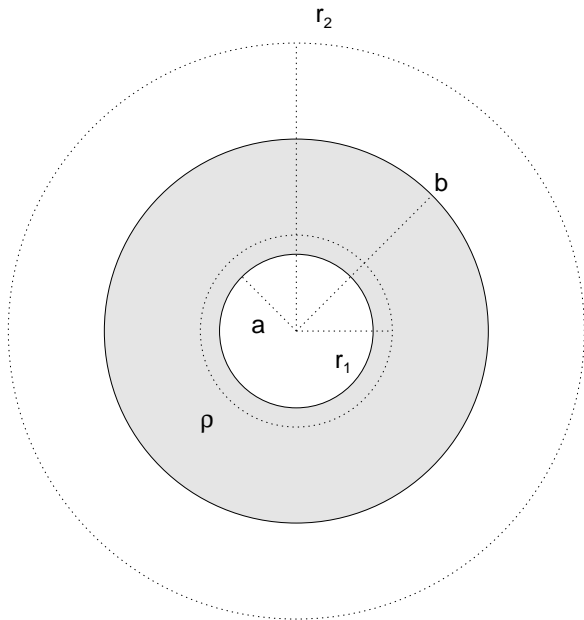
$$q_{inS} = A\sigma$$

we recover our previous hard-won result

$$|\mathbf{E}| = \frac{\sigma A}{2A\epsilon_0} = \frac{\sigma}{2\epsilon_0}$$



6.1 Review Problems



72

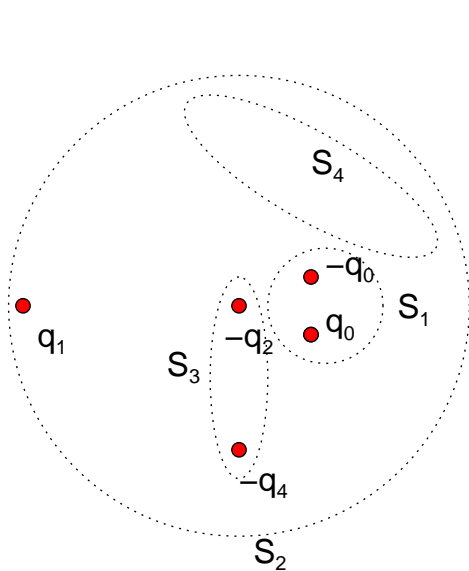
A hollow ball of inner radius a and outer radius b is made of charged dust of density ρ . Compute the electric field strength for $a < r_1 < b$. and for the exterior region, $r_2 > b$

73

Suppose that a voltage function in the first quadrant of the xy -plane is

$$V(x, y) = \frac{V_0}{b^2}(x^2 - y^2)$$

in which V_0 is a voltage and b is a constant with units of length. Compute the components of the electric field, sketch the equipotentials and field lines.



74

Each dotted curve in the figure to the left is a closed surface in cross section. Compute the electric flux through each of them.

75

Suppose that a voltage function in the first quadrant of the xy -plane is

$$V(x, y) = \frac{V_0}{b^2}(x^2 + y^2), \quad V_0, b \text{ constant}$$

Compute the components of the electric field, sketch the equipotentials and field lines.

76

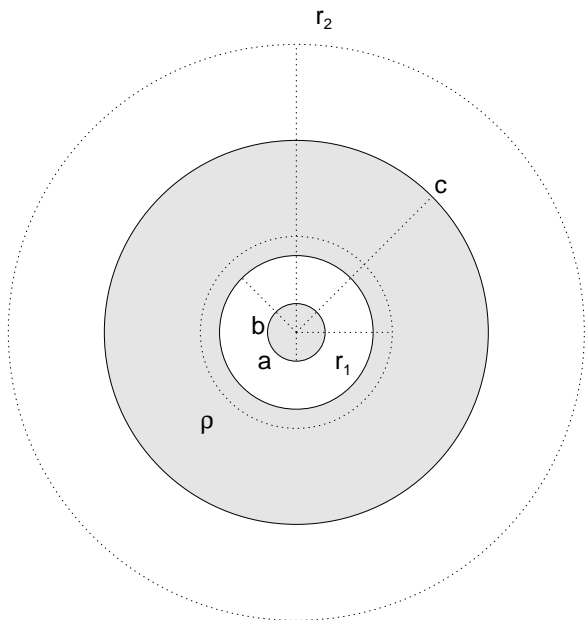
Consider the problem of building up a solid ball of charged dust of density ρ and radius R by depositing successive layers of charged dust on the surface by bringing the material in from infinity where the voltage is zero. Show that when the radius is r , the work needed to increase the radius to $r + dr$ by applying more dust is

$$dW = \frac{4\pi}{3\epsilon_0} \rho^2 r^4 dr$$

and compute the total work needed to build up the sphere to a final charge Q and radius R .

77

Suppose that the potential in the xy -plane is given by $V(x, y) = \frac{V_0}{b^2}xy$. Compute the equations of equipotentials and electric field lines and sketch them in the first quadrant.



78

A hollow ball of inner radius b and outer radius c is made of charged dust of density ρ contains an inner solid ball of the same matter of radius a . Compute the electric field strength for $b < r_1 < c$, and for the exterior region, $r_2 > c$

79

Let $E_x = -ax$ and $E_y = ay$, where a is a constant. Sketch the electric field lines. Can you find a voltage function for this field?

80

The voltage function in the xy plane is given by the expression

$$V(x, y) = \frac{V_0}{\pi} \tan^{-1} \frac{y}{x}$$

A. Find the equation of the zero-voltage equipotential, and the $V = \frac{V_0}{4}$ equipotential.

B. Compute E_x and E_y .

81

A region of space around the origin contains an electric field $\mathbf{E} = 4.0 \frac{N}{m \cdot C} x \mathbf{i}$. Such a field cannot exist in empty space. Find the total charge within a cube of side $a = 0.5 m$ centered on the origin, with its six faces possessing normals in the six cardinal directions $\pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}$.

82

Compute the flux of the electric field $\mathbf{E} = 5.0 \frac{N}{C} \mathbf{k}$ through the top hemisphere of the sphere

$$x^2 + y^2 + z^2 = (2\text{ m})^2, \quad z \geq 0$$

It will help tremendously to draw a picture.

83

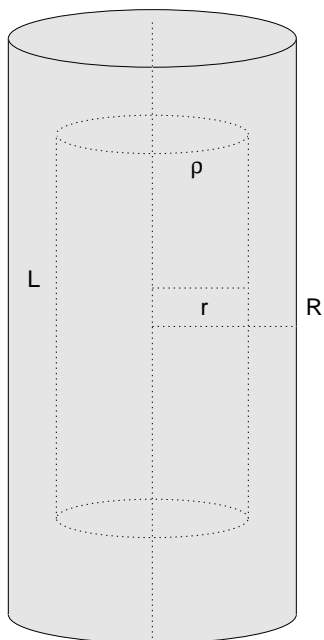
Compute the flux of the electric field $\mathbf{E} = 5.0 \frac{N}{C} \mathbf{i}$ through the right half of the cylindrical surface

$$x^2 + y^2 = (2.0\text{ m})^2, \quad 0 \leq z \leq 3.0\text{ m}, \quad x \geq 0$$

It will help tremendously to draw a picture.

84

Use Gauss' law to compute the electric field at a distance $r < R$ from the central axis of an infinitely long cylinder made of charged dust of density ρ , with radius R . Compute the field strength for some $r > R$.

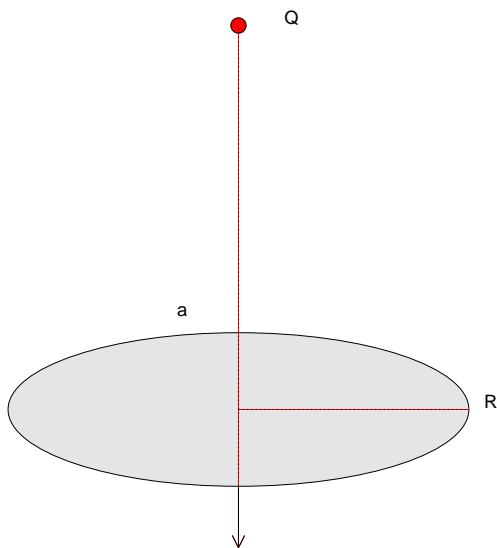


85

Compute the flux of the electric field of a point charge q at the center of a cube of side a through one of the faces of the cube (the top face).

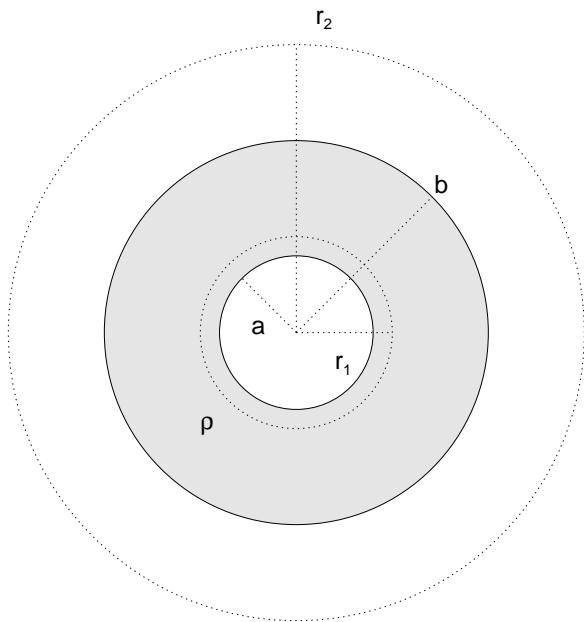
86

Compute the flux of the electric field $\mathbf{E} = 5.0 \frac{N}{C \cdot m} x \mathbf{k} + 2.0 \frac{N}{C \cdot m^2} z^2 \mathbf{i}$ through a square lying in the xy -plane with corners at $(1.0\text{ m}, 1.0, m)$, $(-1.0\text{ m}, 1.0, m)$, $(1.0\text{ m}, -1.0, m)$, $(-1.0\text{ m}, -1.0, m)$.



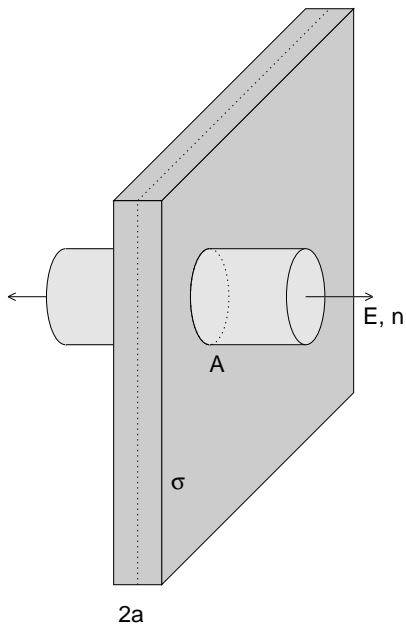
87

Compute the flux of the field of a charge Q through a disk of radius R placed a perpendicular distance a from the charge. The normal to the disk points downward (in the field direction).



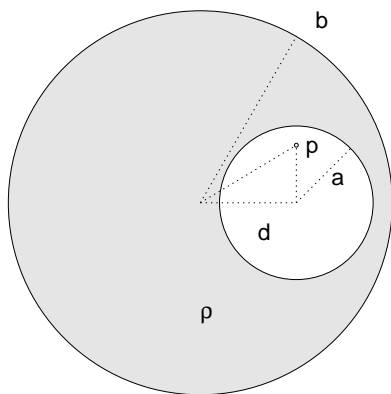
88

An infinitely long cylinder of outer radius b and inner radius a **seen in cross-section** has the region between a and b filled with charged dust of density ρ . Find $\mathbf{E}(\mathbf{r})$ for $\mathbf{r} = r_{1,2}$ with $a \leq r_1 \leq b$ and $r_2 > b$.



89

Compute the electric field strength at a point a distance x (measured perpendicularly) from the median plane (dotted) of an infinite planar slab of charged dust of density ρ and thickness $2a$, for both cases $x < a$ and $x > a$.



90

A solid ball of charged dust of radius b and density ρ has an empty spherical hole inside of radius a , whose center is a distance d from the center of the ball. Find the electric field at a point $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ within the hollow.