Bargmann-Wigner construction

What is the connection between Hamiltons's σ_i "quaternions" and angular momentum? Recall that Hamilton devised a simple algebraic means of combining two rotations into a resultant single rotation, using the gadgets

$$R_{\mathbf{n}}(\theta) = \cos\frac{\theta}{2} + i\boldsymbol{\sigma}\cdot\mathbf{n}\,\sin\frac{\theta}{2}$$

where $\mathbf{n}$ is the vector of direction cosines of the axis of rotation. In Hamilton's way of performing rotations, we associate a matrix $\mathbf{r}\cdot\boldsymbol{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$ with a vector $\mathbf{r}$, and rotation about axis $\mathbf{n}$ by angle θ transforms vector $\mathbf{r}$ into $\mathbf{r}'$;

$$\mathbf{r}'\cdot\boldsymbol{\sigma} = R^\dagger_{\mathbf{n}}(\theta)\cdot\Big(\mathbf{r}\cdot\boldsymbol{\sigma}\Big)\cdot R_{\mathbf{n}}(\theta)$$

The beauty of this approach is that if we rotate a body by angle θ about axis $\mathbf{n}$, and apply a second rotation by ϕ about axis $\mathbf{n}'$, this is equivalent to rotating just once by ψ about $\mathbf{n}''$, and these two items can be found from matrix multiplication

$$R_{\mathbf{n}'}(\phi)\cdot R_{\mathbf{n}}(\theta) = R_{\mathbf{n}''}(\psi)$$

Example. Given $\theta, \mathbf{n}, \phi, \mathbf{n}'$, compute ψ.
Note that

$$2\cos\frac{\psi}{2} = Tr\,R_{\mathbf{n}'}(\phi) = Tr\begin{pmatrix} \cos\frac{\psi}{2} + in_z\sin\frac{\psi}{2} & (n_y+in_x)\sin\frac{\psi}{2} \\ (-n_y+in_x)\sin\frac{\psi}{2} & \cos\frac{\psi}{2} - in_z\sin\frac{\psi}{2} \end{pmatrix}$$

therefore

$$2\cos\frac{\psi}{2} = Tr\Big(R_{\mathbf{n}'}(\phi)\cdot R_{\mathbf{n}}(\theta)\Big)$$

Example. We know that a counterclockwise rotation by θ about the z axis transforms

$$\mathbf{r} = (x,y,z) \to \mathbf{r}' = (x',y',z') = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta, z)$$

or in Hamilton's language, with

$$R_{\mathbf{k}}(\theta) = \cos\frac{\theta}{2} + i\sigma_z\sin\frac{\theta}{2} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$$

$$\begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} = \begin{pmatrix} z & e^{-i\theta}(x-iy) \\ e^{i\theta}(x+iy) & -z \end{pmatrix}$$

$$= \begin{pmatrix} z & (x\cos\theta - y\sin\theta) - i(y\cos\theta + x\sin\theta) \\ (x\cos\theta - y\sin\theta) + i(y\cos\theta + x\sin\theta) & -z \end{pmatrix}$$

What we normally think of as a vector is in fact a tensor (a matrix) in Hamilton's formalism. So what is a vector in Hamilton's formalism? It would be the object that would transform as

$$s' = R_{\mathbf{n}}(\theta)\cdot s, \qquad \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = R_{\mathbf{n}}(\theta)\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

which we call a spinor. The space of spinors has two basis vectors

$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

from which we can create "adjoint" basis vectors

$$\uparrow^\dagger = (1, 0), \qquad \downarrow^\dagger = (0, 1)$$

and from these gadgets one can create "spinor-tensors" such as

$$\uparrow \otimes \uparrow = \begin{pmatrix} 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \uparrow \otimes \downarrow = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\downarrow \otimes \uparrow = \begin{pmatrix} 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \downarrow \otimes \downarrow = \begin{pmatrix} 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

or

$$\uparrow^\dagger \otimes \uparrow = \Big(1\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \uparrow^\dagger \otimes \downarrow = \Big(1\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\downarrow^\dagger \otimes \uparrow = \Big(0\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \downarrow^\dagger \otimes \downarrow = \Big(0\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and so forth, making it very clear that a vector (in the conventional sense) is composed of two spinors

$$\mathbf{r} \cdot \boldsymbol{\sigma} = z\Big(\uparrow^\dagger \otimes \uparrow - \downarrow^\dagger \otimes \downarrow \Big) + (x - iy) \Big(\downarrow^\dagger \otimes \uparrow \Big) + (x - iy) \Big(\uparrow^\dagger \otimes \downarrow \Big)$$

The connection with angular momentum. We have established that both classically and quantum mechanically, angular momentum generates rotations. The infinitismal generator of rotations of state vectors are the quantum angular momentum operators;

$$\Big(1 + \frac{i}{\hbar}\theta L_z\Big) \Psi(x, y, z) = \Big(1 + \theta(x\partial_y - y\partial_x)\Big) \Psi(x, y, z)$$

$$= \Psi(x, y, z) + \theta x\frac{\partial \Psi}{\partial y} - \theta y\frac{\partial \Psi}{\partial x} = \Psi(x - y\theta, y + x\theta, z) \approx \Psi(x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta, z)$$

and so in the case of the quantum state vector being a spinor

$$R_{\mathbf{k}}(\theta) \cdot s = e^{i\theta\frac{1}{2}\sigma_z} \cdot s = e^{i\frac{\theta}{\hbar}s_z} \cdot s = s'$$

allowing us to associate the spin angular momentum operator (acting on spinors) with a sigma-matrix

$$s_x = \tfrac{1}{2}\hbar\sigma_x, \qquad s_y = \tfrac{1}{2}\hbar\sigma_y, \qquad s_z = \tfrac{1}{2}\hbar\sigma_z$$

and explicit calculation with these matrices confims that they obey the algebraic relations of angular momentum operators

$$[s_x, s_y] = i\hbar s_z, \qquad [s_y, s_z] = i\hbar s_x, \qquad [s_z, s_x] = i\hbar s_y$$

Because the s_i operators are 2×2 matrices, we say that the matrices are a two-dimensional representation ρ of the algebra of operators

$$\rho(s_x) = \tfrac{1}{2}\hbar\sigma_x = \begin{pmatrix} 0 & \frac{1}{2}\hbar \\ \frac{1}{2}\hbar & 0 \end{pmatrix}, \quad \rho(s_y) = \tfrac{1}{2}\hbar\sigma_y = \begin{pmatrix} 0 & -i\frac{1}{2}\hbar \\ i\frac{1}{2}\hbar & 0 \end{pmatrix}, \quad \rho(s_z) = \tfrac{1}{2}\hbar\sigma_z = \begin{pmatrix} \frac{1}{2}\hbar & 0 \\ 0 & -\frac{1}{2}\hbar \end{pmatrix}$$

It is very typical of physicists to leave off the ρ, and to essentally equate the operators with the matrices representing them. I will do this from now on.

Construction of higher dimensional representations of an algebra

The Bargmann-Wigner construction produces higher dimensional representations of the angular momentum operator algebra

$$[s_x, s_y] = i\hbar s_z, \qquad [s_y, s_z] = i\hbar s_x, \qquad [s_z, s_x] = i\hbar s_y$$

from the lowest dimensinal faithful representation (the two-dimensional) by reducing tensor products to irreducible summands.

Stage zero. The spin-$\frac{1}{2}$ fundamental spinors are

$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2}, \tfrac{1}{2}\rangle, \qquad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

so named by specifying the eigenvalues of two operators

$$s_z \uparrow = \tfrac{1}{2}\hbar \uparrow, \qquad s_z \downarrow = -\tfrac{1}{2}\hbar \downarrow$$

and

$$\mathbf{s}^2 = (s_x)^2 + (s_y)^2 + (s_z)^2 = 3\tfrac{1}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for which $[s_z, \mathbf{s}^2] = 0$, so the two operators possess similtaneous eigenvectors;

$$\mathbf{s}^2 \uparrow = \frac{3}{4}\hbar^2 \uparrow, \qquad \mathbf{s}^2 \downarrow = \frac{3}{4}\hbar^2 \downarrow$$

We will find that when acting on a spin ℓ state vector, that

$$\mathbf{s}^2 \psi_\ell = \hbar^2 \ell(\ell+1)\psi_\ell$$

and since $\frac{3}{4} = \frac{1}{2}(\frac{1}{2}+1)$, we obtain the two labels used to specify ↑ and ↓ in ket notation.

Some basic rules for spinors and tensors; the projection of one spinor $s = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ onto another $s' = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ is the "dot product"

$$s^\dagger \cdot s' = (\alpha^*, \beta^*) \cdot \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \alpha^*\gamma + \beta^*\delta$$

or in ket notation $\langle s|s'\rangle$. This can be extended to tensors

$$(s \otimes s')^\dagger \cdot (r \otimes r') = (s^\dagger \cdot r)(s'^\dagger \cdot r')$$

The action of a tensor operator $A \otimes B$ on a tensor $s \otimes r$ is

$$A \otimes B(s \otimes r) = (As) \otimes (Br)$$

The tensor product of two tensors is a higher tensor

$$(a \otimes b \otimes c) \otimes (d \otimes e) = a \otimes b \otimes c \otimes d \otimes e)$$

Note that tensor products are order-dependent; let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$;

$$A \otimes B = \begin{pmatrix} a\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & b\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ c\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & d\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix}$$

$$B \otimes A = \begin{pmatrix} \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{pmatrix}$$

Stage one. Construction of "spin-1" representations. We start by building all possible two-particle wavefunctions, using the idea that the phase space variables of the two particles are completely independent of one another. This means that two-particle state vectors are tensors such as

$$v_1 = \uparrow \otimes \uparrow \equiv \uparrow\uparrow, \qquad v_2 = \uparrow \otimes \downarrow \equiv \uparrow\downarrow$$

$$v_3 = \downarrow \otimes \uparrow \equiv \downarrow\uparrow, \qquad v_4 = \downarrow \otimes \downarrow \equiv \downarrow\downarrow$$

In the notation $x \otimes y \otimes z$, we mean that particle one is in state "x", particle two in state "y", and particle three in state "z", and so forth.

The next step is to demonstrate that each such two-particle state is the eigenstate of a total spin z-component operator

$$\Sigma_z = s_{z,1} + s_{z,2} = s_z \otimes 1 + 1 \otimes s_z$$

(we extend the action of s_z to an action on tensors, s_i is a linear operator on the vector space spanned by $\uparrow, \downarrow$),

$$\Sigma_z \uparrow\uparrow = (s_z \uparrow) \otimes \uparrow + \uparrow \otimes (s_z \uparrow) = \frac{\hbar}{2}\Big(\uparrow\uparrow + \uparrow\uparrow \Big) = \hbar \uparrow\uparrow$$

$$\Sigma_z \uparrow\downarrow = (s_z \uparrow) \otimes \downarrow + \uparrow \otimes (s_z \downarrow) = \frac{\hbar}{2}\Big(\uparrow\downarrow - \uparrow\downarrow \Big) = 0 \uparrow\downarrow$$

$$\Sigma_z \downarrow\uparrow = (s_z \downarrow) \otimes \uparrow + \downarrow \otimes (s_z \uparrow) = \frac{\hbar}{2}\Big(- \downarrow\uparrow + \downarrow\uparrow \Big) = 0 \downarrow\uparrow$$

$$\Sigma_z \downarrow\downarrow = (s_z \downarrow) \otimes \downarrow + \downarrow \otimes (s_z \downarrow) = -\frac{\hbar}{2}\Big(\downarrow\downarrow + \downarrow\downarrow \Big) = -\hbar \downarrow\downarrow$$

The next step is to pick out the **highest-weight vector** (the one with the biggest Σ_z eigenvalue) from the bunch, and apply the lowering operator to it repeatedly.
Recall that

$$[s_z, s_x \pm i s_y] = \pm(s_x \pm i s_y), \qquad [s_z, s_\pm] = \pm s_\pm$$

and for the two dimensional (spin-$\frac{1}{2}$) case (lets let $\hbar = 1$)

$$s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s_- \uparrow = \downarrow, \qquad s_- \downarrow = 0$$

$$s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_+ \uparrow = 0, \qquad s_+ \downarrow = \uparrow$$

Extend the action of $s_\pm$ to tensors; acting on rank two tensors

$$\Sigma_\pm = \sigma_\pm \otimes 1 + 1 \otimes \sigma_\pm$$

or on rank three

$$\Sigma_\pm = \sigma_\pm \otimes 1 \otimes 1 + 1 \otimes \sigma_\pm \otimes 1 + 1 \otimes 1 \otimes \sigma_\pm$$

and so forth.

Of our four tensors, the state $\uparrow\uparrow$ is highest weight. This is the top of the chain;

$$\Sigma_- \uparrow\uparrow = (s_- \uparrow)\uparrow + \uparrow(s_- \uparrow) = \downarrow\uparrow + \uparrow\downarrow$$

Normalize this new state

$$v_2' = \frac{\downarrow\uparrow + \uparrow\downarrow}{\sqrt{2}}$$

so that

$$\Sigma_- v_1 = \sqrt{2}\, v_2$$

Apply the lowering operator

$$\Sigma_- v_2' = \frac{(s_- \downarrow)\uparrow + (s_- \uparrow)\downarrow + \downarrow(s_- \uparrow) + \uparrow(s_- \downarrow)}{\sqrt{2}} = \sqrt{2}\ \downarrow\downarrow = \sqrt{2}\, v_4$$

but the norm of v_4 is already one (it is normalized). Try to apply the lowering operator again

$$\Sigma_- \downarrow\downarrow = 0$$

therefore the three normalized vectors $\{v_1, v_2', v_4\}$ span a three dimensional vector space, and form an orthonormal basis of it, in which we can make a 3×3 matrix representation of the angular momentum algebra. Remember that all vector spaces of the same dimension are isomorphic, so call

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

note that

$$\Sigma_z(av_1 + bv_2' + cv_4) = (av_1 - cv_4), \qquad \Sigma_z \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix}$$

so

$$\rho(\Sigma_z) = \Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Sigma_-(av_1 + bv_2' + cv_4) = (\sqrt{2}av_2' + \sqrt{2}bv_4), \qquad \Sigma_- \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}a \\ \sqrt{2}b \end{pmatrix}$$

so

$$\rho(\Sigma_-) = \Sigma_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

Similarly

$$\rho(\Sigma_+) = \Sigma_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

(show that $[\Sigma_+, \Sigma_-] = 2\Sigma_z$ as we expect). Now construct

$$\boldsymbol{\Sigma}^2 = (\Sigma_x)^2 + (\Sigma_y)^2 + (\Sigma_z)^2 = \tfrac{1}{2}\Big(\Sigma_+\Sigma_- + \Sigma_-\Sigma_+\Big) + (\Sigma_z)^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and compute

$$\mathbf{\Sigma}^2 v_1 = 2v_1 = 1(1+1)v_1, \qquad \mathbf{\Sigma}^2 v_2' = 2v_2' = 1(1+1)v_2', \qquad \mathbf{\Sigma}^2 v_4 = 2v_4 = 1(1+1)v_4$$

and so we call

$$v_1 = \uparrow\uparrow = |1,1\rangle, \qquad v_2' = \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}} = |1,0\rangle, \qquad v_4 = \downarrow\downarrow = |1,-1\rangle$$

We have accounted for three out of four of the tensor product state vectors, to account for the last ones we apply Gram-Schmidt orthogonalization; from v_1, v_2, v_3, v_4 make a unit vector orthogonal to v_1, v_2', v_4. The only possibility is

$$v_3' = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}$$

note that

$$\Sigma_\pm v_3' = 0, \qquad \Sigma_z v_3' = 0, \qquad \boldsymbol{\sigma}^2 v_3' = 0$$

and so

$$v_3' = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} = |0,0\rangle$$

this state is a scalar, it represents a state with no angular momentum, made of two particles each with half-integer angular momentum.
The decomposition of tensor products to irreducible representations of dimension three and one is often written as

$$\mathbf{\tfrac{1}{2}} \otimes \mathbf{\tfrac{1}{2}} = \mathbf{1} \oplus \mathbf{0}$$

Spin-2 states. We can iterate the process and make spin-2 states by starting with tensors

$$v_1 \otimes v_1 = v_1 v_1, \quad v_1 \otimes v_2' = v_1 v_2', \quad v_1 v_4$$

$$v_2' \otimes v_1 = v_2' v_1, \quad v_2' \otimes v_2' = v_2' v_2', \quad v_2' v_4$$

$$v_4 v_1, \quad v_4 v_2', \quad v_4 v_4$$

and apply Σ_z;

$$\Sigma_z v_1 v_1 = (\Sigma_z v_1) v_1 + v_1 (\Sigma_z v_1) = 2 v_1 v_1, \qquad \Sigma_z v_2' v_1 = (\Sigma_z v_2') v_1 + v_2' (\Sigma_z v_1) = v_2' v_1, \qquad \Sigma_z v_4 v_1 = 0$$

$$\Sigma_z v_1 v_2' = v_1 v_2', \qquad \Sigma_z v_1 v_4 = 0, \qquad \Sigma_z v_2' v_2' = 0, \qquad \Sigma_z v_2' v_4 = -v_2' v_4, \qquad \Sigma_z v_4 v_2' = -v_4 v_2', \qquad \Sigma_z v_4 v_4 = -2 v_4 v_4$$

Start with the highest weight vector $v_1 v_1$;

$$\Sigma_- v_1 v_1 = \sqrt{2} v_2' v_1 + \sqrt{2}\, v_1 v_2' = 2\Big(\frac{v_2' v_1 + v_1 v_2'}{\sqrt{2}}\Big)$$

$$\Sigma_-\Big(\frac{v_2' v_1 + v_1 v_2'}{\sqrt{2}}\Big) = \Big(\frac{\sqrt{2} v_4 v_1 + \sqrt{2} v_2' v_2' + \sqrt{2} v_2' v_2' + \sqrt{2} v_1 v_4}{\sqrt{2}}\Big) = \sqrt{6}\Big(\frac{v_1 v_4 + 2 v_2' v_2' + v_4 v_1}{\sqrt{6}}\Big)$$

$$\Sigma_-\Big(\frac{v_1 v_4 + 2 v_2' v_2' + v_4 v_1}{\sqrt{6}}\Big) = \Big(\frac{\sqrt{2} v_2' v_4 + 2\sqrt{2} v_4 v_2' + 2\sqrt{2} v_2' v_4 + \sqrt{2} v_4 v_2'}{\sqrt{6}}\Big) = \sqrt{6}\Big(\frac{v_2' v_4 + v_4 v_2'}{\sqrt{2}}\Big)$$

$$\Sigma_-\Big(\frac{v_2' v_4 + v_4 v_2'}{\sqrt{2}}\Big) = 2 v_4 v_4$$

If we call

$$w1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \Big(\frac{v_2'v_1 + v_1v_2'}{\sqrt{2}}\Big) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_3 = \Big(\frac{v_1v_4 + 2v_2'v_2' + v_4v_1}{\sqrt{6}}\Big) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$w_4 = \Big(\frac{v_2'v_4 + v_4v_2'}{\sqrt{2}}\Big) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad w_5 = v_4v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

then

$$\Sigma_z \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 2a \\ b \\ 0 \\ -d \\ -2e \end{pmatrix}, \qquad \rho(\Sigma_z) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\Sigma_- \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 2a \\ \sqrt{6}b \\ \sqrt{6}c \\ 2d \end{pmatrix}, \qquad \rho(\Sigma_-) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \qquad \rho(\Sigma_+) = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we fid that $\mathbf{\Sigma}^2 w_i = 6w_i = 2(2+1)w_i$, so we equate

$$w_1 = |2,2\rangle, \quad w_2 = |2,1\rangle, \quad w_3 = |2,0\rangle, \quad w_4 = |2,-1\rangle, \quad w_5 = |2,-2\rangle$$

We have accounted for five out of nine of the tensor states, what about the others?
The process can be applied to the remainder. Note that **in any representation the highest weight vector is unique**. We found that there were two tensors with Σ_z eigenvalue equal to one,

$$\Sigma_z v_2'v_1 = v_2'v_1, \qquad \Sigma_z v_1v_2' = v_1v_2'$$

and the vector $w_2 = \frac{v_2'v_1 + v_1v_2'}{\sqrt{2}}$ belongs to the spin-2 manifold. Apply Gram-Schmidt to create a vector $u_1 = \frac{v_2'v_1 - v_1v_2'}{\sqrt{2}}$ that is orthogonal to w_2, and note that $\Sigma_z u_1 = u_1$. Apply Σ_- to this

$$\Sigma_-\Big(\frac{v_2'v_1 - v_1v_2'}{\sqrt{2}}\Big) = \sqrt{2}\Big(\frac{v_4v_1 - v_1v_4}{\sqrt{2}}\Big)$$

Note that this vector is also $\perp$ to w_i, $i = 1,2,3,4,5$! Apply the lowering operator again

$$\Sigma_-\Big(\frac{v_4v_1 - v_1v_4}{\sqrt{2}}\Big) = \sqrt{2}\Big(\frac{v_4v_2' - v_2'v_4}{\sqrt{2}}\Big)$$

and this vector is also $\perp$ to w_i, $i = 1,2,3,4,5$**, and to** u_1 **and** $u_2 = \Big(\frac{v_4v_1 - v_1v_4}{\sqrt{2}}\Big)$! Note that

$$\Sigma_-\Big(\frac{v_4v_2' - v_2'v_4}{\sqrt{2}}\Big) = 0$$

Call

$$u_1 = \frac{v_2'v_1 - v_1v_2'}{\sqrt{2}}, \quad u_2 = \Big(\frac{v_4v_1 - v_1v_4}{\sqrt{2}}\Big), \quad u_3 = \Big(\frac{v_4v_2' - v_2'v_4}{\sqrt{2}}\Big)$$

then

$$\Sigma_z u_1 = u_1, \qquad \Sigma_z u_2 = 0, \qquad \Sigma_z u_3 = -u_3$$

$$\Sigma_- u_1 = \sqrt{2}\, u_2, \qquad \Sigma_- u_2 = \sqrt{2}\, u_3, \qquad \Sigma_- u_3 = 0$$

so that $\{u_1 = |1,1\rangle, u_2 = |1,0\rangle, u_3 = |1,-1\rangle\}$ **form a seperate set of spin-1 states**.
This accounts for eight out of nine of the tensors. We had three tensors that were zero eigenvalues

$$\Sigma_z v_2'v_2' = 0, \qquad \Sigma_z v_1v_4 = 0, \qquad \Sigma_z v_4v_1 = 0$$

and we have two states of known angular momentum that are also zero eigenvectors of Σ_z

$$|2,0\rangle = w_3 = \Big(\frac{v_1v_4 + 2v_2'v_2' + v_4v_1}{\sqrt{6}}\Big), \qquad |1,0\rangle = u_2 = u_2 = \Big(\frac{v_4v_1 - v_1v_4}{\sqrt{2}}\Big)$$

so apply Gram-Schmidt to the set $\{v_2'v_2', v_1v_4, v_4v_1\}$ to create a vector $\perp$ to $|2,0\rangle$ and to $|1,0\rangle$;

$$|0,0\rangle = \Big(\frac{v_1v_4 - v_2'v_2' + v_4v_1}{\sqrt{3}}\Big)$$

and you should be able to show that

$$\mathbf{\Sigma}^2\Big(\frac{v_1v_4 - v_2'v_2' + v_4v_1}{\sqrt{3}}\Big) = 0, \qquad \Sigma_z\Big(\frac{v_1v_4 - v_2'v_2' + v_4v_1}{\sqrt{3}}\Big) = 0$$

You will see this particular decomposition of tensors written as $\mathbf{1} \otimes \mathbf{1} = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$, the general rule for decomposition being

$$\ell \otimes \ell' = (\ell + \ell') \oplus (\ell + \ell' - \mathbf{1}) \oplus \cdots \oplus |\ell - \ell'|$$

General matrix elements
Let

$$s_-|\ell,\ell\rangle = a(\ell)\,|\ell,\ell-1\rangle$$

We know that

$$s_-|1,1\rangle = \sqrt{2}\,|1,0\rangle$$

Note that $|\ell,\ell\rangle \otimes |1,1\rangle$ is the highest weight state in the tensor manifold $\boldsymbol{\ell} \otimes \mathbf{1}$;

$$\Sigma_-|\ell,\ell\rangle \otimes |1,1\rangle = \Sigma_-|\ell+1,\ell+1\rangle = a(\ell)\,|\ell,\ell-1\rangle \otimes |1,1\rangle + \sqrt{2}|\ell,\ell\rangle \otimes |1,0\rangle = a(\ell+1)\,|\ell+1,\ell\rangle$$

normalize;

$$\Sigma_-|\ell+1,\ell+1\rangle = \sqrt{a^2(\ell)+2}\Big(\frac{a(\ell)\,|\ell,\ell-1\rangle \otimes |1,1\rangle + \sqrt{2}|\ell,\ell\rangle \otimes |1,0\rangle}{\sqrt{a^2+2}}\Big)$$

so that

$$a(\ell+1) = \sqrt{a^2(\ell)+2}$$

This says that $a^2(\ell)$ is a polynomial in ℓ no higher that quadratic; $a(\ell) = \alpha\ell^2 + \beta\ell + \delta$

$$\Big(\alpha(\ell+1)^2 + \beta(\ell+1) + \delta\Big) - \Big(\alpha\ell^2 + \beta\ell + \delta\Big) = 2, \qquad \alpha = 0, \quad \beta = 2$$

and from the initial condition $a(1) = \sqrt{2}$ we find that

$$\Sigma_-|\ell,\ell\rangle = \sqrt{2\ell}\,|\ell,\ell-1\rangle$$

Apply the process again, let

$$\Sigma_-|\ell,\ell-1\rangle = b(\ell)\,|\ell,\ell-2\rangle$$

$$\Sigma_-|\ell+1,\ell\rangle = b(\ell+1)\,|\ell+1,\ell-1\rangle = \Sigma_-\Big(\frac{\sqrt{2\ell}\,|\ell,\ell-1\rangle\otimes|1,1\rangle + \sqrt{2}|\ell,\ell\rangle\otimes|1,0\rangle}{\sqrt{2\ell+2}}\Big)$$

$$= \Big(\frac{\sqrt{2\ell}\,b(\ell)\,|\ell,\ell-2\rangle\otimes|1,1\rangle + 2\sqrt{2\cdot 2\ell}\,|\ell,\ell-1\rangle\otimes|1,0\rangle + \sqrt{2\cdot 2}|\ell,\ell\rangle\otimes|1,-1\rangle}{\sqrt{2\ell+2}}\Big)$$

normalize

$$(2\ell+2)b^2(\ell+1) = 2\ell\,b^2(\ell) + 16\ell + 4$$

once again saying that $b(\ell)$ is no more than linear in ℓ, $b(\ell) = \alpha\ell+\beta$;

$$(2(\ell+1)(\alpha\ell+\alpha+\beta) = 2\ell(\alpha\ell+\beta) + 16\ell + 4, \qquad \alpha = 4, \quad \beta = -2$$

$$\Sigma_-|\ell,\ell-1\rangle = \sqrt{4\ell-2}\,|\ell,\ell-2\rangle$$

Hypothesize that

$$\Sigma_-|\ell,m\rangle = f(\ell,m)\,|\ell,m-1\rangle$$

with $f(\ell,m)$ no worse than quadratic in ℓ, m;

$$f^2(\ell,m) = a\ell^2 + b\ell m + cm^2 + d\ell + em + f$$

Because $\Sigma_-|\ell,-\ell\rangle = 0$, we see that

$$(a+c-b)\ell^2 + (d-e)\ell + f = 0, \qquad b = (a+c), \qquad d = e, \qquad f = 0$$

leaving only two three free parameters a, c, d

$$f^2(\ell,m) = a\ell^2 + (a+c)\ell m + cm^2 + d(\ell+m)$$

Use what we have learned about $a(\ell) = f(\ell,\ell)$ and $b(\ell) = f(\ell,\ell-1)$;

$$2\ell = 2(a+c)\ell^2 + 2d\ell, \qquad \Longrightarrow \qquad a+c = 0, \quad d = 1$$

$$4\ell - 2 = a\ell^2 - a(\ell-1)^2 + (\ell+\ell-1) \qquad \Longrightarrow \qquad a = 1$$

This results in the general matrix element formula ($\hbar = 1$)

$$\Sigma_-|\ell,m\rangle = \sqrt{\ell^2 - m^2 + (\ell+m)}\,|\ell,m-1\rangle = \sqrt{(\ell+m)(\ell-m+1)}\,|\ell,m-1\rangle$$

and of course

$$\Sigma_z|\ell,m\rangle = m\,|\ell,m\rangle, \qquad \boldsymbol{\Sigma}^2|\ell,m\rangle = \ell(\ell+1)\,|\ell,m\rangle,$$