

# 1 Simple elements of group theory

A group  $G$  is a set of objects together with a combination or “multiplication” law under which the set is closed, together with a few other restrictions. The set of properties are

1. Closure;  $g_1, g_2 \in G$  implies  $g_1 g_2 \in G$  and  $g_2 g_1 \in G$ .
2. There is a unique identity element  $e \in G$  such that for any  $g \in G$  we have  $g e = g$  and  $e g = g$ .
3. For each  $g \in G$  there is a unique inverse  $g^{-1}$  such that  $g g^{-1} = g^{-1} g = e$ .
4. The “multiplication” or composition rule is associative  $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ .

It is remarkable that these few properties result in so much structure. The most common occurrences of groups in physics is as transformations of vectors or wavefunctions. For example all volume-preserving and orientation preserving linear transformations of a vector space form the **orthogonal group** on that vector space. The set of all local unitary transformations  $\psi \rightarrow \psi' = U(x)\psi$  that leave the action of a Dirac fermion interacting with an electromagnetic field form the **gauge group**.

Group theory forms a large and complete discipline within mathematics, and is worthy of study in its own right. Physicists often simply need to know what the **representations** of a group are. These are sets of matrices that under matrix multiplication obey exactly the same multiplication table that the group elements do. We will study group theory with a focus on representation theory.

A table of all binary multiplication rules for a group is called its **multiplication table**, and it is very useful but can be tedious or downright impractical to obtain in practice. For low-order groups it is an easy job. The **order**  $N_G$  of a group is the number of group elements.

For example there is only one group of order 2;  $G = \{e, a\}$  with  $a a = e$ . The multiplication table is;

$G_2$	e	a
e	e	a
a	a	e

One can readily enumerate all groups of order three  $\{e, a, b\}$  by constructing all possible multiplication tables. There is again only one

$G_3$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

This provides us with an example of an **abelian** or commutative group, one for which  $g g' = g' g$  for all  $g, g' \in G$ .

We begin to see some action at order four;

$G_4$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$G'_4$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

These two groups are clearly different because the multiplication tables are different.

**A subgroup**  $H$  of  $G$  is a non-empty subset of  $G$  such that

$$a, b \in H \text{ implies } ab \in H, \quad a \in H \text{ implies } a^{-1} \in H$$

The multiplication table readily shows subgroups, we see that  $G_3$  has no subgroups, neither does  $G_4$  but  $G'_4$  contains  $G_2$  as a subgroup.

Subgroups are important for several reasons. Let  $H$  be a subgroup of  $G$  with  $a$  and  $b \in G$ , if

$ab^{-1} \in H$  we say that  $a = b \text{ mod } H$ , meaning that there exists some  $h \in H$  such that  $a = hb$ . This is an **equivalence relation** in that first  $a = a \text{ mod } H$  since  $e \in H$ ,  $a = b \text{ mod } H$  implies  $b = a \text{ mod } H$  since  $a = hb$  means  $h^{-1}a = b$  and so  $ba^{-1} = h^{-1} \in H$ , and most importantly  $a = b \text{ mod } H$  and  $b = c \text{ mod } H$  implies  $a = c \text{ mod } H$ . Why? note

$$a = hb, \quad b = h'c, \quad (ab^{-1})(bc^{-1}) = hh' = h'' \in H$$

and so  $ac^{-1} = h''$ , or  $a = h''c$ . We say  $b \in [a]$  if  $b = a \text{ mod } H$

This equivalence relation is responsible for the decomposition of  $G$  into a union of disjoint sets (called **right cosets** of  $H$  in  $G$ ). The notation is

$$Ha = \{ha \mid h \in H\}$$

and the property responsible for this partitioning is

$$Ha = [a] = \{x \in G \mid a = x \text{ mod } H\}$$

Two right cosets are either the same or are disjoint (no common elements). First note that  $Ha \subset [a]$  since any  $g \in Ha$  is of the form  $g = ha$  for some  $h \in H$ , and so  $h^{-1} = ag^{-1}$  making  $ag^{-1} \in H$ , or  $a = g \text{ mod } H$ . This places  $g \in [a]$ . It is equally easy to show  $[a] \subset Ha$ , and so  $[a] = Ha$ .

It is clear that there is a one-to-one correspondence between elements of two distinct cosets  $Hg_1$  and  $Hg_2$ ; pick some  $h \in H$ , then the correspondence is  $hg_1 \leftrightarrow hg_2$ , which is one-to-one since  $hg_1 = h'g_1$  would imply  $h^{-1}h' = e$  within  $H$ , but  $H$  is a group and so inverses are unique, forcing  $h = h'$ .

What we have is a partition of  $G$  into disjoint sets, each of size  $N_H$  since  $He = H$  is a right coset. This implies Lagrange's theorem

**The order  $N_H$  of  $H$  a subgroup of  $G$  is a divisor of  $N_G$ .** This tells us right away that a group of order four can have subgroups of orders one, two or four only.

The **index** of  $H$  in  $G$  is the number of right cosets of  $G$  by  $H$ ;

$$i_G(H) = N_G/N_H$$

Suppose that  $G$  has two subgroups  $H_1$  and  $H_2$ , then the set  $H_1H_2$  is the set of all products of the form  $h_1h_2$  with  $h_1 \in H_1$  and  $h_2 \in H_2$ . This may or may not be a subgroup of  $G$ .

If  $H_1H_2 = H_2H_1$ , then  $H_1H_2$  is a subgroup of  $G$ . Note that a priori these two sets need not be the same.

Since clearly  $H_1H_2 \subseteq G$  we need to establish that  $H_1H_2$  is a group.

**Closure.**  $h_1h_2, h'_1h'_2$  are in  $H_1H_2$ , but if  $H_1H_2 = H_2H_1$  then there exists  $h''_1 \in H_1$  and  $h''_2 \in H_2$  such that  $h_2h'_1 = h''_1h''_2$  so that

$$(h_1h_2)(h'_1h'_2) = h_1(h_2h'_1)h'_2 = h_1(h''_1h''_2)h'_2 = (h_1h''_1)(h''_2h'_2) \in H_1H_2$$

**Inverse.** Let  $h_1h_2 \in H_1H_2$ , then  $(h_1h_2)^{-1} = h_2^{-1}h_1^{-1} \in H_2H_1 = H_1H_2$  so  $(h_1h_2)^{-1} \in H_1H_2$  and  $H_1H_2$  is a group within  $G$ , a subgroup of  $G$ .

We can show that if  $H_1$  and  $H_2$  are any subgroups of  $G$ , the size of the set  $H_1H_2$  is

$$N_{H_1H_2} = \frac{N_{H_1}N_{H_2}}{N_{H_1 \cap H_2}}$$

The most important of all subgroups are the **normal subgroups**;  $N$  a subgroup of  $G$  is normal (write  $N \triangleleft G$ ) if for every  $g \in G$  and  $n \in N$  we have  $gng^{-1} \in N$ . Right and left cosets of a normal subgroup coincide.

Note that  $N \triangleleft G$  if and only if  $gNg^{-1} = N$  for all  $g \in G$ . The forward proof is trivial, since if  $gNg^{-1} = N$  then  $gNg^{-1} \subset N$ . Now imagine that  $N \triangleleft G$ , then  $gNg^{-1} \subset N$ , and  $g^{-1}Ng \subset N$ , so

$$g(g^{-1}Ng)g^{-1} \subset g(N)g^{-1}, \quad \text{therefore} \quad N \subset gNg^{-1}$$

forcing  $gNg^{-1} = N$ . If  $N \triangleleft G$  then  $gNg^{-1} = N$ , so  $gNg^{-1}g = Ng$  or  $gN = Ng$  for any  $g \in G$  making left and right cosets coincide.

If  $g_1$  and  $g_2$  are related by  $g_1 = gg_2g^{-1}$  for some  $g \in G$  we say that  $g_1 \sim g_2$  (they are **conjugate**, and conjugacy is another equivalence relation that will partition the group into disjoint subsets. This partition is critical in representation theory. The set of elements  $g_1 \sim g$  is called the **conjugacy class** of  $g$ , or the **orbit** of  $g$ .)

Here is the best part of the normal subgroup story;

$$(Na)(Nb) = N(aN)b = N(Na)b = N(ab)$$

and so the cosets themselves have the group property of closure.

We call  $G/N$  the set of right cosets of  $N \triangleleft G$  in  $G$ , a typical member of the set can be denoted by  $Ng$ . This set is closed under multiplication, it has an “identity” coset (in fact its  $N = Ne$ ) and each coset  $Ng$  has an “inverse”  $(Ng)^{-1} = g^{-1}N = Ng^{-1}$ . Furthermore the multiplication is associative

$$(Na Nb)Nc = (Nab)Nc = N(abc) = Na(Nb Nc) = Na(Nbc) = N(abc)$$

and so the set of cosets of a normal subgroup form the **quotient group** of  $G$  by  $N$ , and

$$N_{G/N} = N_G/N_N$$

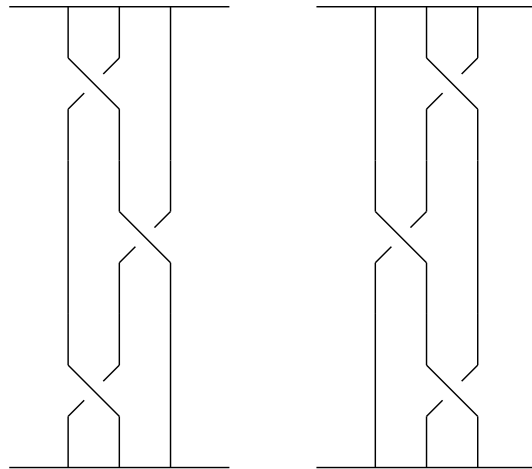
A few more technical introductions. For any  $g \in G$ , the **normalizer**  $N(g)$  of  $g$  in  $G$  is the set  $\{g' \in G \mid g'g = gg'\}$ . The **center**  $Z(G)$  of  $G$  is the set of all  $g \in G$  for which  $gg' = g'g$  for all  $g' \in G$  (the set of elements that commute with all group elements). Both the normalizer of any element and the center of a group  $G$  are subgroups of  $G$ .

### A complete example.

#### The Dirac string trick.

Normally a braid group generated by  $\{g_1, \dots, g_n\}$  with  $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$  (left figure illustrates  $g_1 g_2 g_1 = g_2 g_1 g_2$ ) would be infinite order, since  $g_i^n \neq e$  for any  $n$ , unless other relations are imposed.

Why is it called a “braid” group? Consider  $n$  parallel strings, fixed at a top bar, that can be braided. Let  $g_i = (i \ i + 1)$  represent the act of passing string  $i$  **over** string  $i + 1$ . Then  $g_i^2$  is the act of winding strings  $i$  and  $i + 1$  around each other. The group element  $g_i^{-1}$  passes string  $i + 1$  **over** string  $i$ , so no intertwining occurs in the group element  $g_i g_i^{-1} = e$



If the “bottom bar” of the braid is finite length, one can work the strings around the lower bar ends, or rotate the bar by  $\pi$  radians.

This allows the condition  $g_1g_2g_2g_1 = e$  (right figure) to be imposed on a three-strand braid. **Construct the full group.**

This makes the braid group (in this case) finite; call  $a = g_1$ ,  $b = g_2$ ,  $aba = bab$ , then  $abba = e$  implies  $b^2 = a^{-2}$

$$\begin{aligned} b^4 &= ba^{-2}b = baa^{-4}ab = bab^4ab \\ &= (bab)b^2(bab) = (aba)b^2(aba) \\ &= (ab)(ab^2a)(ba) = (ab)(ba) = e \end{aligned}$$

and

$$a^4 = e, \quad a^2 = a^{-2} = b^2, \quad a^{-1} = a^3, \quad b^{-1} = b^3$$

We begin by constructing words

$$\{e, a, b, ab, ba, a^2 = b^2, a^3, b^3, aba = bab\}$$

and products that cannot be further reduced, made from these words, such as

$$a^3b, ba^3,$$

all other products being reducible, by use of the constraint, except for  $ba^3b = b^3ab = a^3ba = aba^3 = bab^3 = ab^3a = a^3ba$  Of all of these I choose the elements

$$G = \{e, a, b, ab, ba, a^2, aba, a^3, b^3, a^3b, ba^3, ba^3b\}$$

and construct the multiplication table.

$G$	$e$	$a$	$b$	$ab$	$ba$	$a^3$	$b^3$	$aba$	$a^2$	$a^3b$	$ba^3$	$ba^3b$
$e$	$e$	$a$	$b$	$ab$	$ba$	$a^3$	$b^3$	$aba$	$a^2$	$a^3b$	$ba^3$	$ba^3b$
$a$	$a$	$a^2$	$ab$	$b^3$	$aba$	$e$	$a^3b$	$ba^3$	$a^3$	$b$	$ba^3b$	$ba$
$b$	$b$	$ba$	$a^2$	$aba$	$a^3$	$ba^3$	$e$	$a^3b$	$b^3$	$ba^3b$	$a$	$ab$
$ab$	$ab$	$aba$	$a^3$	$ba^3$	$e$	$ba^3b$	$a$	$b$	$a^3b$	$ba$	$a^2$	$b^3$
$ba$	$ba$	$b^3$	$aba$	$e$	$a^3b$	$b$	$ba^3b$	$a$	$ba^3$	$a^2$	$ab$	$a^3$
$a^3$	$a^3$	$e$	$a^3b$	$b$	$ba^3b$	$a^2$	$ab$	$ba$	$a$	$b^3$	$aba$	$ba^3$
$b^3$	$b^3$	$ba^3$	$e$	$ba^3b$	$a$	$ba$	$b^2$	$ab$	$b$	$aba$	$a^3$	$a^3b$
$aba$	$aba$	$ab^3$	$ba^3$	$a$	$b$	$ab$	$ba$	$a^2$	$ba^3b$	$a^3$	$b^3$	$e$
$a^2$	$a^2$	$a^3$	$b^3$	$a^3b$	$ba^3$	$a$	$b$	$ba^3b$	$e$	$ab$	$ba$	$aba$
$a^3b$	$a^3b$	$ba^3b$	$a$	$ba$	$a^2$	$aba$	$a^3$	$b^3$	$ab$	$ba^3$	$e$	$b$
$ba^3$	$ba^3$	$b$	$ba^3b$	$a^2$	$ab$	$b^3$	$aba$	$a^3$	$ba$	$e$	$a^3b$	$a$
$ba^3b$	$ba^3b$	$ab$	$ba$	$a^3$	$b^3$	$a^3b$	$ba^3$	$e$	$aba$	$a$	$b$	$a^2$

From this one can see that  $e$  and  $a^2$  are in their own conjugacy classes, another class is  $\{ab, ba\}$  and  $\{a^3b, ba^3\}$ . The orbit equations

$$bab^{-1} = Z, \quad ba = Zb, \quad \text{has solutions } Z = ba^3b, b, a$$

and none of the remaining elements are self-conjugate and so the conjugacy class structure (orbits) are

$$\{e\}, \{a^2\}, \{ab, ba\}, \{a^3b, ba^3\}, \{a, b, ba^3b\}, \{aba, a^3, b^3\}$$

## Problems

The permutation group on 3 objects is

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3), \quad g_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3), \quad g_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)(2)$$
$$g_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(23), \quad g_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Work out the multiplication table, find all subgroups, indicate any that are normal, and determine the conjugacy classes.

### 1.1 Representation theory

A (matrix) representation of a group is a set of matrices whose matrix multiplication laws preserve the group multiplication table. Consider the group  $G_2 = \{e, a \mid a^2 = e\}$ . A matrix representation is

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and another is

$$\rho'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho'(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They are in fact equivalent, there is a transformation matrix  $C$  such that

$$C\rho(e)C^{-1} = \rho'(e), \quad C\rho(a)C^{-1} = \rho'(a)$$

A dead giveaway is the **trace** or **character** of the representation matrices being the same. The matrices representing all members of a conjugacy class will have the same trace (trace is a class function), note

$$\text{Tr}(\rho(e)) = \text{Tr}(\rho'(e)) = 2, \quad \text{Tr}(\rho(a)) = \text{Tr}(\rho'(a)) = 0$$

Characters were introduced by Frobenius, and are the trace of a matrix representation of a group element

$$\chi_\rho(g) = \text{Tr} \rho(g)$$

and these quantities can be used to determine how many **irreducible** representations a group can have.

**Proof that equivalent representations have same characters.** Two representations  $\rho$  and  $\rho'$  are equivalent if there exists a transformation  $C$  such that

$$\rho'(g) = C\rho(g)C^{-1}, \quad \text{Tr}\rho'(g) = \text{Tr}C\rho(g)C^{-1} = \text{Tr}\rho(g)$$

therefore character is a class function (we assume that  $C = \rho(g)$  for some  $g \in G$ ); if  $g_1$  and  $g_2$  are in the same class then there is a  $g$  such that  $g_2 = gg_1g^{-1}$  so

$$\text{Tr}\rho(g_2) = \text{Tr}\rho(g)\rho(g_1)\rho(g^{-1}) = \text{Tr}\rho(g_1)$$

Let  $\rho(g)$  and  $\rho'(g)$  be irreducible representations of dimensions  $d$  and  $d'$  (so  $\rho(g)$  is a  $d \times d$  matrix with elements  $(\rho(g))_{ij}$ ) respectively. Pick any  $X$  an  $d \times d'$  matrix, and construct the matrix

$$C = \sum_{g \in G} \rho(g) X \rho'(g^{-1}), \quad C_{ij} = \sum_{g \in G} (\rho(g))_{ik} X_{kl} (\rho'(g^{-1}))_{lj},$$

Then

$$\rho(h)C = \sum_{g \in G} \rho(h)\rho(g) X \rho'(g) = \sum_{g \in G} \rho(hg) X \rho'(g) = \sum_{g' \in G} \rho(g') X \rho'(g'^{-1}h) = C\rho'(h)$$

**Schur's lemma** Says that if  $C$  is a constant matrix for which  $\rho(h)C = C\rho'(h)$  for all  $h \in G$  then if  $\rho$  and  $\rho'$  are inequivalent representations,  $C = 0$ . Otherwise  $C^{-1}$  exists (so  $C$  is a square matrix,  $d = d'$ ) and they are clearly equivalent. This lemma also says that the only matrix that commutes with all group matrices is a multiple of the unit matrix. The  $d \neq d'$  case should be clear; that for  $\rho, \rho'$  inequivalent we have  $C = 0$  (since  $\rho(h)C$  is  $d'$ -dimensional and  $C\rho'(h)$  is  $d$ -dimensional). For  $\rho$  and  $\rho'$  equivalent then  $C = \lambda \cdot \rho(e)$ . If we take  $X_{ik} = 0$  and  $X_{j\ell} = 1$  for  $(i, k) \neq (j, \ell)$  we get

$$\sum_{g \in G} (\rho(g))_{ij} (\rho'(g^{-1}))_{\ell k} = 0, \quad \sum_{g \in G} (\rho(g))_{ij} (\rho(g^{-1}))_{\ell k} = \eta_{j\ell} \delta_{ik}$$

for some set of numbers  $\eta_{j\ell}$ . Let  $k = i$  and sum on  $i$ ;

$$\sum_{g \in G} \sum_i (\rho(g))_{ij} (\rho(g^{-1}))_{\ell i} = \eta_{j\ell} \delta_{ii} = d \eta_{j\ell}$$

but

$$\sum_{g \in G} \sum_i (\rho(g))_{ij} (\rho(g^{-1}))_{\ell i} = \sum_{g \in G} \rho(gg^{-1})_{\ell j} = N_G (\rho(e))_{\ell j}$$

and therefore

$$\eta_{j\ell} = \frac{N_G}{d} (\rho(e))_{\ell j} = \frac{N_G}{d} \delta_{j\ell}$$

and we get the group orthogonality relation

$$\sum_{g \in G} (\rho(g))_{ij} (\rho(g^{-1}))_{\ell k} = \frac{N_G}{d} \delta_{ik} \delta_{j\ell}$$

Now set  $i = j$  and  $h = k$  and sum over  $i$  and  $k$  defining the **character** to be the trace of the representation matrix  $\chi_\rho(g) = \text{Tr} \rho(g) = \sum_i (\rho(g))_{ii} = (\rho(g))_{ii}$ ;

$$\sum_{g \in G} \chi_\rho(g) \chi'_\rho(g^{-1}) = 0, \quad \sum_{g \in G} \chi_\rho(g) \chi_\rho(g^{-1}) = \frac{N_G}{d} \text{Tr} \rho(e) = N_G$$

**This is the master formula**, and we can simplify it by establishing some facts about the character that can be elucidated from group theory principles. First of all, for any group element  $g \in G$  the set of powers  $\{g^0 = e, g, g^2, g^3, \dots, g^{N_G}\}$  is a subgroup of  $G$ . Closure and associativity are obvious. If each power is a distinct element, we are in trouble since there are  $N_G + 1 > N_G$  objects in this set. Therefore it must be true that there are redundancies, and there is some power  $n$ , the **order** of  $g$ , for which  $g^n = e$ . The resulting group (since you can see that inverses are also powers of  $g$ )  $\{e, g, g^2, \dots, g^{n-1}\}$  is called a **cyclic** group. Note that  $G_3$  was a cyclic group, that  $b = a^2$ . Consider then that  $g^{N_G} = e$  for any group element, and examine a  $d$ -dimensional representation;

$$(\rho(g))^{N_G} = \rho(g^{N_G}) = \rho(e) = \mathbf{1}_{d \times d}$$

the last equality is true since  $\rho(e)$  commutes with all elements of the group. Now let  $U\rho(g)U^\dagger = \Lambda$  where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_d \end{pmatrix}, \quad \chi_\rho(g) = \text{Tr} \rho(g) = \text{Tr} (U\rho(g)U^\dagger) = \lambda_1 + \lambda_2 + \dots + \lambda_d$$

Note then that

$$(U\rho(g)U^\dagger)^{N_G} = \begin{pmatrix} \lambda_1^{N_G} & 0 & \cdots & 0 \\ 0 & \lambda_2^{N_G} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \lambda_d^{N_G} \end{pmatrix} = \rho(e) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

and so **each term in the character is a root of unity**,

$$\lambda_k = e^{2\pi i \frac{k'}{N_G}}, \quad \text{for some integer } k'$$

Therefore we find that  $\rho(g^{-1}) = \rho^{-1}(g) = \rho^\dagger(g)$  and so  $\chi(g^{-1}) = \overline{\chi(g)} = \chi^*(g)$  and the master character formulas become

$$\sum_{g \in G} \chi_\rho(g) \chi_\rho^*(g) = 0, \quad \sum_{g \in G} \chi_\rho(g) \chi_\rho(g^{-1}) = \sum_{g \in G} \chi_\rho(g) \chi_\rho^*(g) = N_G.$$

**Character is a class function.** The proof is very simple. Let  $g$  and  $g'$  be in the same conjugacy class, then there is some  $h \in G$  such that  $g' = hgh^{-1}$ , and this is honored by any representation  $\rho(g') = \rho(h) \rho(g) \rho(h^{-1})$ , so take traces

$$\chi_\rho(g') = \text{Tr}(\rho(h) \rho(g) \rho(h^{-1})) = \text{Tr}(\rho(h^{-1}) \rho(h) \rho(g)) = \text{Tr}(\rho(g)) = \chi_\rho(g)$$

Let  $G$  have classes  $c$ , each of size  $N(c)$ , then

$$\sum_c N(c) = N_G$$

and each  $g \in c = [g]$  has the same character in representation  $\chi_\rho(c)$ , so that the main character formulas (we call these the **orthogonality rules**) are

$$\sum_c N(c) \chi_\rho(c) \chi_\rho^*(c) = 0, \quad \sum_c N(c) \chi_\rho(c) \chi_\rho^*(c) = N_G.$$

**Every group has a trivial representation** with each element represented by the number 1. The character of every class is 1. This representation is typically denoted by “A”. How many **irreducible** representations does a group have? It has as many irreducible representations as it has classes, since the master character equation is essentially a statement that the characters, regarded as vectors (with their numbers of components equal to the number of classes) are orthogonal to one another.

To see how the details of the irreducible representations are constructed, return to the Dirac braid example, and begin looking for another one dimensional representation ( $\chi_B(e) = 1$ ).

Irrep	{e}	{a <sup>2</sup> }	{ab, ba}	{a <sup>3</sup> b, ba <sup>3</sup> }	{a, b, ba <sup>3</sup> b}	{aba, a <sup>3</sup> , b <sup>3</sup> }
A	1	1	1	1	1	1
B	1	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$

for which the master character formula says

$$1 + \chi_1 + 2\chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 = 0$$

and the group product rules require that

$$\chi_1\chi_2 = \chi_3, \quad \chi_2\chi_3 = \chi_1 = \chi_2, \quad \chi_1\chi_3 = \chi_1 = \chi_2, \quad \chi_1\chi_4 = \chi_1 = \chi_5$$

$$\chi_4\chi_3 = \chi_4 \Rightarrow \chi_3 = 1, \quad \chi_4\chi_4 = \chi_1$$

There are several possibilities; if  $\chi_1 = -1$  then

$$\chi_2 = \chi_1 = -1, \quad \chi_4 = -\chi_5, \chi_4^2 = -1, \chi_4 = \pm i, \quad \chi_5 = \mp i$$

and if  $\chi_1 = 1$ ;

$$\chi_2 = \chi_1 = 1, \quad \chi_4 = \chi_5, \quad 1 + 1 + 2 + 2 + 3\chi_4 + 4\chi_4 = 0 \Rightarrow \chi_4 = -1$$

and so we can construct a total of four one-dimensional representations

Irrep	$\{e\}$	$\{a^2\}$	$\{ab, ba\}$	$\{a^3b, ba^3\}$	$\{a, b, ba^3b\}$	$\{aba, a^3, b^3\}$
A	1	1	1	1	1	1
B	1	-1	-1	1	$i$	$-i$
C	1	-1	-1	1	$-i$	$i$
D	1	1	1	1	-1	-1

This cannot be the whole story, to find the remaining representations we construct the **regular representation**, which a group always has, but is reducible in that it is the direct sum of irreducible representations.

### The group algebra

We can construct a vector space from the group, by regarding the group elements as a basis of some sort. Consider the objects

$$x = a_1g_1 + a_2g_2 + \cdots + a_Ng_N, \quad a_i \in C, \quad g_i \in G$$

These objects can be added;

$$y = \sum_i b_i g_i, \quad x = \sum_i a_i g_i, \quad x + y = \sum_i (a + b_i) g_i$$

and multiplied, since  $gg_i = g_j$ , and so form an algebra, which is also a vector space. But the so-formed vector space is **reducible**.

Any group or algebra has a very special representation, called the **regular** representation of the group, or **adjoint** representation of the algebra, defined by the multiplication table. We think of  $g g_1 = g_2$  as saying that “ $g$  acts as a linear transformation on  $g_1$  transforming it into  $g_2$ ”. Since linear transformations are transformations on vector spaces, this interpretation is made legitimate within the context of the group algebra.

**Example** consider  $G_3 = \{e, a, b \mid ab = ba\}$ , then

$$ee = e, ea = a, eb = b \quad \text{implies} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ e \\ b \end{pmatrix} = \begin{pmatrix} a \\ e \\ b \end{pmatrix}$$

so on the basis  $G$  of the group algebra (vector space) we find  $\rho_{reg}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , and

$$ae = a, aa = b, ab = e \quad \text{implies} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ e \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \\ e \end{pmatrix}$$

$$\rho_{reg}(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{reg}(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Calculate the character of this representation,  $G$  has three classes

$$\chi_{reg}(e) = Tr \rho_{reg}(e) = 3, \quad \chi_{reg}(a) = Tr \rho_{reg}(a) = 0, \quad \chi_{reg}(b) = Tr \rho_{reg}(b) = 0$$

**The structure of the character table, with each group element appearing once and only once in every row and column** guarantees that the only group element for which  $(\rho_{reg}(g))_{ii} \neq 0$  is  $g = e$ , so that  $\chi_{reg}(e) = N_G$  and  $\chi_{reg}(g \neq e) = 0$  for any group.

So what is meant by **reducible and irreducible**? A representation is reducible if the vector space contains subspaces that actually form representations of  $G$  themselves, and irreducible if the vector space contains no such subspaces. Lets construct an example. Since  $G_3$  is so small, we can find sets of complex numbers that honor the group multiplication table, and so form one-dimensional representations.

$G_3$	e	a	b
A	1	1	1
B	1	x	y

$G_3$	e	a	b
A	1	1	1
B	1	$e^{2\pi i \frac{1}{3}}$	$e^{2\pi i \frac{2}{3}}$
C	1	$e^{2\pi i \frac{2}{3}}$	$e^{2\pi i \frac{1}{3}}$

Begin with a class separation and the trivial representation, and propose a second representation; the orthogonality character formula requires  $1 \cdot 1 + 1 \cdot x + 1 \cdot y = 0$ , and since  $x$  and  $y$  are complex numbers that honor the group multiplication table we must have  $\rho(a)\rho(a) = \chi(a)\chi(a) = \chi(b)$  or  $x^2 = y$  and  $\rho(a)\rho(b) = \chi(a)\chi(b) = \chi(e)$  or  $xy = 1$ , which requires  $x^3 = 1$  or  $x = e^{2\pi i \frac{1}{3}}$  or  $x = e^{2\pi i \frac{2}{3}}$ , which in turn makes  $y = e^{2\pi i \frac{2}{3}}$  or  $y = e^{2\pi i \frac{1}{3}}$  respectively. The character table has two more irreducible representations (a one dimensional vector space has no proper subspaces).

**I propose that the regular representation of  $G_3$  has three subspaces, one forming an A representation, one transforming as B and one transforming as C.** This can be proved with the group character orthogonality formula, which is a major application of this formula. Suppose some representation is a direct sum of vector spaces each forming an irreducible representation of  $G$

$$\rho = \rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_r$$

or possibly containing multiple copies

$$\rho = \bigoplus_i n_i \rho_i, \quad \text{then} \quad \chi_\rho(g) = \sum_i n_i \chi_{\rho_i}(g)$$

where  $\{\rho_i\}$  are the set of irreducible representations. Pick some irreducible representation  $\rho_j$  and multiply

$$\chi_\rho(g) \chi_{\rho_j}^*(g) = \sum_i n_i \chi_{\rho_i}(g) \chi_{\rho_j}^*(g)$$

and sum over  $g \in G$

$$\sum_{g \in G} \chi_\rho(g) \chi_{\rho_j}^*(g) = \sum_{g \in G} \sum_i n_i \chi_{\rho_i}(g) \chi_{\rho_j}^*(g) = \sum_i n_i \delta_{ij} = n_j N_G$$

and "Parseval's" identity

$$\sum_{g \in G} \chi_\rho(g) \chi_\rho^*(g) = \sum_i \sum_j \sum_{g \in G} n_i n_j \chi_{\rho_i}(g) \chi_{\rho_j}^*(g) = \sum_{i,j} N_G n_i n_j \delta_{i,j} = N_G \sum_i n_i^2$$

If we do this for the regular representation of  $G_3$  we find that

$$n_A = 1, \quad n_B = 1, \quad n_C = 1, \quad \rho_{reg} = \rho_A \oplus \rho_B \oplus \rho_C$$

**Let the dimension** of each irreducible representation  $\rho_i$  of  $G$  be  $d_i$ , and use the fact that for any  $G$   $\chi_{reg}(e) = N_G$  and  $\chi_{reg}(g \neq e) = 0$ . The regular representation is reducible, and so

$$\chi_{reg}(g) = \sum_i n_i \chi_i(g)$$

and use orthogonality to compute  $n_i$ , the multiplicity of representation  $\rho_i$  in the regular representation;

$$n_i = \frac{1}{N_G} \sum_g \sum_i \chi_{reg}(g) \chi_i(g^{-1}) = \frac{1}{N_G} \sum_g \sum_i N_G \delta_{g,e} \chi_i(g^{-1}) = \chi_i(e) = d_i$$

**The regular representation contains  $d_i = \dim(\rho_i)$  copies of each irreducible representation of  $G$ ,  $\rho_{reg}(g) = \sum_i d_i \rho_i(g)$ .** In this formula, set  $g = e$  and take traces to get the **character dimension formula**

$$\rho_{reg}(e) = \sum_i d_i \rho_i(e), \quad \chi_{reg}(e) = N_G = \sum_i d_i \chi_i(e) = \sum_i d_i^2$$

**Continue the analysis of the Dirac braid group**, since apparently there are more representations. Apply the dimension rule;  $1 + 1 + 1 + 1 + d_E^2 + d_F^2 = 12$ , so  $d_E = d_F = 2$ , and so

Irrep	$\{e\}$	$\{a^2\}$	$\{ab, ba\}$	$\{a^3b, ba^3\}$	$\{a, b, ba^3b\}$	$\{aba, a^3, b^3\}$
A	1	1	1	1	1	1
B	1	-1	-1	1	$i$	$-i$
C	1	-1	-1	1	$-i$	$i$
D	1	1	1	1	-1	-1
E/F	2	$\psi_1$	$\psi_2$	$\psi_3$	$\psi_4$	$\psi_5$

the orthogonality rule with  $A$  and  $D$  say

$$2 + \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 3\psi_5 = 0, \quad 2 + \psi_1 + 2\psi_2 + 2\psi_3 - 3\psi_4 - 3\psi_5 = 0$$

so

$$2 + \psi_1 + 2\psi_2 + 2\psi_3 = 0, \quad \psi_4 = -\psi_5$$

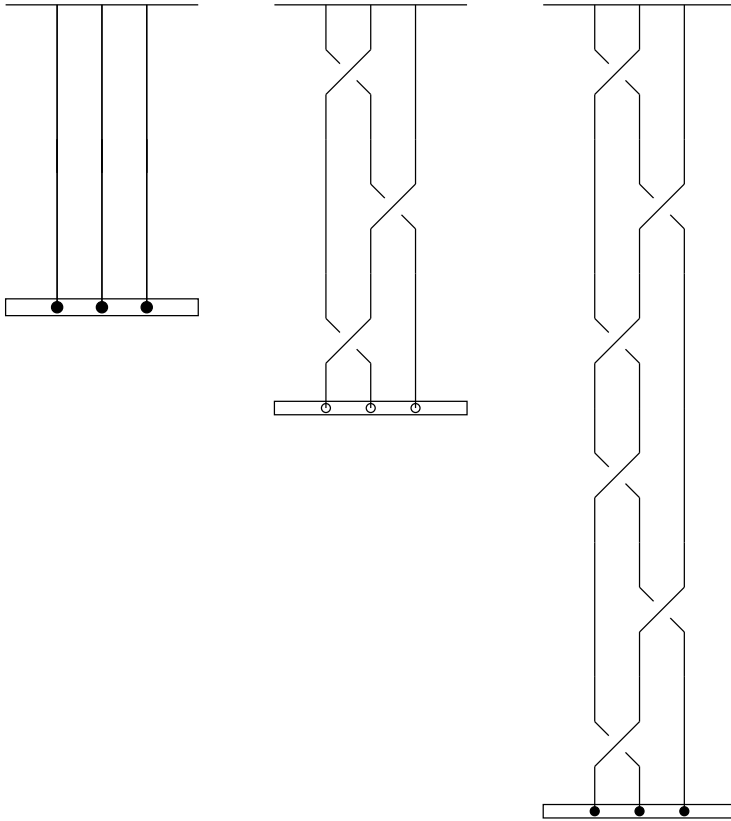
the orthogonality rule with  $B$  and  $C$  say

$$2 - \psi_1 - 2\psi_2 + 2\psi_3 = 0, \quad \psi_4 = \psi_5, \quad \text{compare; } 2 + 2\psi_3 = 0, \psi_3 = -1, \quad \psi_1 + 2\psi_2 = 0$$

resulting in two possible solutions

$$\psi_1 = 2, \psi_2 = -1, \quad \text{or} \quad \psi_1 = -2, \psi_2 = 1$$

Irrep	$\{e\}$	$\{a^2\}$	$\{ab, ba\}$	$\{a^3b, ba^3\}$	$\{a, b, ba^3b\}$	$\{aba, a^3, b^3\}$
A	1	1	1	1	1	1
B	1	-1	-1	1	$i$	$-i$
C	1	-1	-1	1	$-i$	$i$
D	1	1	1	1	-1	-1
E	2	2	-1	-1	0	0
F	2	-2	1	-1	0	0



The subgroup  $H = \{e, a^2\}$  consists of whole classes and so is normal  $gHg^{-1} \in H$ . Furthermore from the table one sees  $a^2g = ga^2, \forall g \in G$  so that  $H$  is the center of  $G$ . The group  $G$  is a double cover of a dihedral group (or of  $S_3$ ), and is a discrete subgroup of  $\text{spin-su}(2)$ .

Twist the bottom bar by  $\pi$  radians, this puts braid into state  $a^{-1}b^{-1}a^{-1}$ , and a  $2\pi$  twist results in

$$\begin{aligned} (a^{-1}b^{-1}a^{-1})^2 &= (a^3b^3a^3)^2 \\ &= (ab)(a^2)(ba) = (aba)^2 = a^2 \neq e \end{aligned}$$

and so a  $4\pi$  twist restores the braid to its untwisted state

$$(a^{-1}b^{-1}a^{-1})^4 = (a^2)^2 = e$$

(the gadget has the spinor property).

## Problems

- 1 Work out the character table for  $G_4$ .
- 2 Work out the character table for  $G'_4$ .
- 3 Work out the character table for the permutation group on three objects.
- 4 Take three objects  $x, y, z$  and form a vector whose components are  $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ . Use it to construct a three dimensional representation of the permutation group on three objects. Is it irreducible? If so, reduce it by finding the bases in the group algebra for the irreducible subspaces.
- 5 Construct the regular representation for the permutation group on three objects. Use the group orthogonality formula to reduce it, and actually do so, find the bases in the group algebra for the irreducible subspaces.

## 1.2 More on the general braid group

The braid group relations are

$$g_i g_j = g_j g_i, \quad |i - j| \geq 2, \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_i^{-1} = g_i^{-1} g_i = e.$$

Even though it is an infinite dimensional group ( $g_i^n \neq e$  for any  $n$ ), it has four **generators!** This means that any group element can be expressed in terms of four fundamental elements. Call  $\Delta = g_1 g_2 g_3 \cdots g_{N-1}$  and  $\Omega = g_1 \Delta$ . Then

$$g_i \Delta = \Delta g_{i-1}, \quad 2 \leq i \leq N-1$$

and

$$g_1 \Delta \Delta = \Delta \Delta g_{N-1}$$

This is instructive to prove;

$$\begin{aligned} \Delta^2 &= (g_1 g_2 g_3 g_4 g_5 \cdots g_{N-1}) (g_1 g_2 g_3 g_4 g_5 \cdots g_{N-1}) = (g_1 g_2 g_1 g_3 g_2 g_4 g_3 g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1}) \\ &= (g_1 g_2 g_1) (g_3 g_2 g_4 g_3 g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1}) = (g_2 g_1 g_2) (g_3 g_2 g_4 g_3 g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1}) \\ &= g_2 g_1 (g_2 g_3 g_2) g_4 g_3 g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1} = g_2 g_1 (g_3 g_2 g_3) g_4 g_3 g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1} \\ &= g_2 g_1 g_3 g_2 (g_3 g_4 g_3) g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1} = g_2 g_1 g_3 g_2 (g_4 g_3 g_4) g_5 g_4 g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1} \\ &= g_2 g_1 g_3 g_2 g_4 g_3 (g_4 g_5 g_4) g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1} = g_2 g_1 g_3 g_2 g_4 g_3 (g_5 g_4 g_5) g_6 g_5 \cdots g_{N-2} g_{N-1} g_{N-2} g_{N-1} \\ &= \cdots = (g_2 g_1) (g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-2} g_{N-1} g_{N-2}) g_{N-1} \\ &= (g_2 g_1) (g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \end{aligned}$$

and so

$$\begin{aligned} g_1 \Delta^2 &= (g_1 g_2 g_1) (g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1 g_2) (g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1) (g_2 g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1) (g_3 g_2 g_3) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1) (g_3 g_2) (g_3 g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1) (g_3 g_2) (g_4 g_3 g_4) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1) (g_3 g_2) (g_4 g_3) (g_4 g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= \cdots = (g_2 g_1) (g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-2} g_{N-1} g_{N-2}) g_{N-1} g_{N-1} \\ &= (g_2 g_1) (g_3 g_2) (g_4 g_3) (g_5 g_4) (g_6 g_5) \cdots (g_{N-1} g_{N-2} g_{N-1}) g_{N-1} g_{N-1} = \Delta^2 g_{N-1} \end{aligned}$$

then with

$$\begin{aligned} g_a &= \Delta g_{a-1} \Delta^{-1} = \Delta^{a-1} g_1 (\Delta^{-1})^{a-1}, \quad g_a \Delta^{a-1} = \Delta^{a-1} g_1 \\ g_{N-1} &= \Delta g_{N-2} \Delta^{-1}, \quad g_{N-1} = \Delta^2 g_{N-3} \Delta^{-2}, \quad \cdots, \quad g_{N-1} = \Delta^{N-1-a} g_a (\Delta^{-1})^{N-1-a} \end{aligned}$$

and use

$$g_a = \Delta^{a-1} g_1 (\Delta^{-1})^{a-1}$$

apply both to

$$g_1 \Delta \Delta = \Delta \Delta g_{N-1}, \quad \Delta^{1-a} g_a \Delta^{a-1} \Delta^2 = \Delta^2 \Delta^{N-1-a} g_a (\Delta^{-1})^{N-1-a}, \quad \Delta^N g_a \Delta^{-N} = g_a$$

It may not be necessary to use  $\Omega$  but you see it in the literature;

$$g_1 = \Omega \Delta^{-1}, \quad g_i = \Delta g_{i-1} \Delta^{-1} = \Delta^2 g_{i-2} (\Delta^{-1})^2 = \Delta^{i-1} g_1 (\Delta^{-1})^{i-1} = \Delta^{i-1} \Omega (\Delta^{-1})^i$$

$$\Delta^N = \Omega^{N-1}$$

The entire group is generated by  $\{\Delta, \Omega, \Delta^{-1}, \Omega^{-1}\}$  (or alternatively by  $\{\Delta, g_1, \Delta^{-1}, g_1^{-1}\}$  ?). Note that if we impose  $g_i = g_i^{-1}$  ( $g_i$  becomes a transposition of strings  $i$  and  $i + 1$ ) then the senses of crossing are indistinguishable and the braid group on  $N$  strings becomes the finite permutation group on  $N$  objects.