Physics 441 Homework I solutions

1. Multiply by $\dot{x}$ and integrate, let the constant of integration be called E

$$m\ddot{x}\dot{x} = \frac{d}{dt}\Big(\tfrac{1}{2}m\dot{x}^2\Big) = -kx\dot{x} = \frac{d}{dt}\Big(-\tfrac{1}{2}kx^2\Big), \qquad \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E$$

$$\dot{x} = \sqrt{\frac{2}{m}(E - \tfrac{1}{2}kx^2)}, \qquad \int_0^x \frac{dx}{\sqrt{\frac{2}{m}(E - \frac{1}{2}kx^2)}} = \int_0^t dt = t$$

Let $x = \sqrt{\frac{2E}{k}}\sin\theta$, $dx = \sqrt{\frac{2E}{k}}\cos\theta\, d\theta$

$$\sqrt{\frac{k}{m}}\,t = \int_0^\theta d\theta = \theta, \qquad x(t) = \sqrt{\frac{2E}{k}}\sin\theta = \sqrt{\frac{2E}{k}}\sin\Big(\sqrt{\frac{k}{m}}t\Big)$$

2. See the animated GIF gotten from using

```
  pl_parampl ("BITMAPSIZE", "300x300");
  pl_parampl ("BG_COLOR", "white");
  pl_parampl ("TRANSPARENT_COLOR", "white");
  pl_parampl ("GIF_ITERATIONS", "100");
  pl_parampl ("GIF_DELAY", "5");
 pl_handle = pl_newpl ("gif", stdin, stdout, stderr);
```

and dumping the output of the run into a file "prob2.gif"

```
gcc SHM.c -L/usr/X11R6/lib -lplot -lXaw -lXmu -lXt -lSM -lICE -lXext -lX11 -lm
./a.out > prob2.gif
```

which you can view in your web-browser at URL **http://rustam.uwp.edu/441/prob2.gif**.

3. You can see that Eq.1.26 is a pair of summable series

$$x(t) = a_0\cos(\omega t) + \frac{a_1}{\omega}\sin(\omega t)$$

so you can solve both problems by picking $a_0 = x_0, a_1 = 0$ and $a_0 = 0, a_1 = v_0$ respectively.

4. Let $L = \sqrt{1+\dot{y}^2}$, $\dot{y} = \frac{dy}{dx}$, then the least action principle states that

$$\frac{d}{dx}\Big(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\Big) = 0, \qquad \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = C, \qquad \dot{y} = \sqrt{\frac{C^2}{1-C^2}}, \quad y = \sqrt{\frac{C^2}{1-C^2}}\,x + C'$$

5. Let $L = \frac{1}{y}\sqrt{1+\dot{y}^2}$, $\dot{y} = \frac{dy}{dx}$, then the least action principle states that

$$\frac{d}{dx}\Big(\frac{\dot{y}}{y\sqrt{1+\dot{y}^2}}\Big) = -\frac{\sqrt{1+\dot{y}^2}}{y^2}, \qquad -\frac{\dot{y}^2}{y^2\sqrt{1+\dot{y}^2}} + \frac{\ddot{y}}{y(1+\dot{y}^2)^{\frac{3}{2}}} = -\frac{\sqrt{1+\dot{y}^2}}{y^2}$$

which simplifies to

$$y\ddot{y} = -1 - \dot{y}^2, \qquad \frac{d}{dx}\Big(y\dot{y}\Big) = -1$$

Integrate once

$$y\dot{y} = \frac{d}{dx}\left(\tfrac{1}{2}y^2\right) = -x + C, \quad \text{and integrate again;} \quad \tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + Cx + C'$$

which describes circles centered on the x-axis!
To perform an arc-length calculation you must **parameterize** the path.
The distance between these points measured along such a circle $x^2 + y^2 = R^2$ from $\theta = \frac{\pi}{4}$ to $\theta = \frac{3\pi}{4}$ is (let $x = R\cos\theta, y = R\sin\theta$)

$$s = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sqrt{dx^2 + dy^2}}{y} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{R\,d\theta}{R\sin\theta} = \ln\tan\frac{\theta}{2}\bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 1.7627471361453$$

now along the line $y = \frac{R}{\sqrt{2}}$, $x \in [-\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}]$, $x = \frac{R}{\sqrt{2}}t$, $-1 \le t \le 1$;

$$s' = \int_{-1}^{1} \frac{\sqrt{0 + \frac{R^2}{2}}\,dt}{\frac{R}{\sqrt{2}}} = 2$$

so the distance really is shorter if we measure along the circle!

6. The time of travel is

$$T = \frac{1}{c}\sqrt{(x_0 + x_1)^2 + y_0^2} + \frac{n}{c}\sqrt{(x_f - x_1)^2 + y_f^2}$$

Extremize with respect to x_1;

$$0 = \frac{1}{c}\frac{(x_0 + x_1)}{\sqrt{(x_0 + x_1)^2 + y_0^2}} - \frac{n}{c}\frac{(x_f - x_1)}{\sqrt{(x_f - x_1)^2 + y_f^2}}$$

but from the figure you can see that this is

$$1\sin\theta_0 = n\sin\theta_f$$

For the second part,

$$E = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_1^2 + V_0, \qquad v_1 = \sqrt{v_0^2 - \frac{2}{m}V_0} = v_0\sqrt{1 - \frac{V_0}{E}} \le v_0$$

Momentum in the y-direction is not conserved because there is a force in the y-direction applied at $y = 0$; $-\frac{\Delta V}{\Delta y} = F_y \neq 0$, but there are no forces applied in the x-direction, so $p_{xi} = p_{xf}$

$$mv_0\sin\theta_0 = mv_1\sin\theta_f, \qquad \sin\theta_0 = \frac{v_1}{v_0}\sin\theta_f = \sqrt{1 - \frac{V_0}{E}}\sin\theta_f$$

which says that $\theta_f > \theta_0$! Measure the bending of light, if it bends towards the normal in a “slow” medium it is a wave, away from the normal in a “slow” medium it is a particle!