Physics 441 Homework III solutions

18.

If you let $\xi = p + i\omega x$, then $\dot{\xi} = \dot{p} + i\omega\dot{x} = -\omega^2 x + i\omega p = i\omega(p + i\omega x) = i\omega\,\xi$, which is trivial to integrate; $\frac{d\xi}{\xi} = i\omega t$, $\ln\frac{\xi(t)}{\xi(0)} = i\omega t$, so

$$p+i\omega x = \xi(t) = \xi(0)\,e^{i\omega t} = (p(0)+i\omega x(0))\,e^{i\omega t} = \Big(p(0)\cos(\omega t) - x(0)\omega\,\sin(\omega t)\Big) + i\omega\Big(x(0)\cos(\omega t) + \frac{p(0)}{\omega}\,\sin(\omega t)\Big)$$

19.

A.

$$p_x = \frac{\partial L}{\dot{x}} = m\dot{x}, \quad \dot{x} = \frac{p_x}{m}, \qquad H = p\dot{x} - L = \frac{1}{2m}p_x^2 + mgx$$

B. Look at Eq.4.39, you must Legendre transform all velocity terms out of L in the same way to get H (well, that is pretty obvious, you must treat all variables equally)

$$p_x = \frac{\partial L}{\dot{x}} = m\dot{x}, \qquad p_y\frac{\partial L}{\dot{y}} = m\dot{y}, \qquad H = p_x\dot{x} - p_y\dot{y} - L = \frac{1}{2m}(p_x^2 + p_y^2) - \frac{mMG}{r}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{mMgx}{(x^2+y^2)^{\frac{3}{2}}}, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{mMgy}{(x^2+y^2)^{\frac{3}{2}}}$$

Polar coordinates; $x = r\cos\phi, y = r\sin\phi$

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mMG}{r}, \quad p_r = \frac{\partial L}{\dot{r}} = m\dot{r}, \quad p_\phi = \frac{\partial L}{\dot{\phi}} = mr^2\dot{\phi}$$

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L = \frac{1}{2m}(p_r^2 + \frac{1}{r^2}p_\phi^2) - \frac{mMG}{r}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{1}{mr^2}p_\phi, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{mMG}{r^2}, \quad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

so p_ϕ is conserved.

C. I will set this one up in polar only.

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \tfrac{1}{2}m\omega^2r^2, \quad p_r = m\dot{r}, \quad p_\phi = mr^2\dot{\phi}$$

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L = \frac{1}{2m}(p_r^2 + \frac{1}{r^2}p_\phi^2) + \tfrac{1}{2}m\omega^2r^2$$

so once again we see that p_ϕ will be a conserved quantity.

D. We actually did this one in class, see the example of the bead constrained to move on a rotating wire in Ch. 3. **This is an important example.**

$$\dot{x}' = (\dot{x}\cos\omega t - \dot{y}\sin\omega t) - \omega(x\sin\omega t + y\cos\omega t), \qquad \dot{y}' = (\dot{y}\cos\omega t + \dot{x}\sin\omega t) - \omega(y\sin\omega t - x\cos\omega t)$$

$$\dot{x}'^2 + \dot{y}'^2 = (\dot{x}^2 + \dot{y}^2) + \omega^2(x^2+y^2) - 2\omega(y\dot{x} - x\dot{y})$$

so that in the rotating frame (temporarily leave out the Ω terms, they are just potentials, nothing special)

$$L = \tfrac{1}{2}m\Big((\dot{x}^2 + \dot{y}^2) + \omega^2(x^2+y^2) - 2\omega(y\dot{x} - x\dot{y})\Big)$$

in which we see ordinary kinetic terms plus centripetal potential plus Coriolis terms.

$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} - \omega y), \quad p_y = \frac{\partial L}{\partial \dot{y}} = m(\dot{y} + \omega x), \qquad \dot{x} = \frac{p_x}{m} + \omega y, \quad \dot{y} = \frac{p_y}{m} - \omega x$$

$$H = p_x(\frac{p_x}{m} + \omega y) + p_y(\frac{p_y}{m} - \omega x) - \tfrac{1}{2} m\Big((\frac{p_x}{m} + \omega y)^2 + (\frac{p_y}{m} - \omega x)^2\Big) + \omega^2(x^2 + y^2) - 2\omega(y(\frac{p_x}{m} + \omega y) - x(\frac{p_y}{m} - \omega x))\Big)$$

which collapses to

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \omega y\, p_x - \omega x\, p_y + \tfrac{1}{2} m \Omega^2 (x^2 + y^2)$$

with the Ω terms restored. The Hamiltonian equations of motion are (with the Ω term back in)

$$\dot{p_x} = \omega p_y - m\Omega^2 x, \qquad \dot{p_y} = -\omega p_x - m\Omega^2 y, \quad \dot{x} = \frac{1}{m} p_x + \omega y, \quad \dot{y} = \frac{1}{m} p_y - \omega x \tag{1}$$

Let $\xi = x + iy$, $\eta = p_x + ip_y$, then

$$\dot{\xi} = \dot{x} + i\dot{y} = \frac{1}{m}(p_x + ip_y) + \omega(y - ix) = \frac{1}{m}\eta - i\omega\xi$$

$$\dot{\eta} = \dot{p}_x + i\dot{p}_y = \omega(p_y - ip_x) - m\Omega^2(x + iy) = -i\omega\eta - m\Omega^2\xi$$

Use what you learned in elementary calculus, introduce an integrating factor

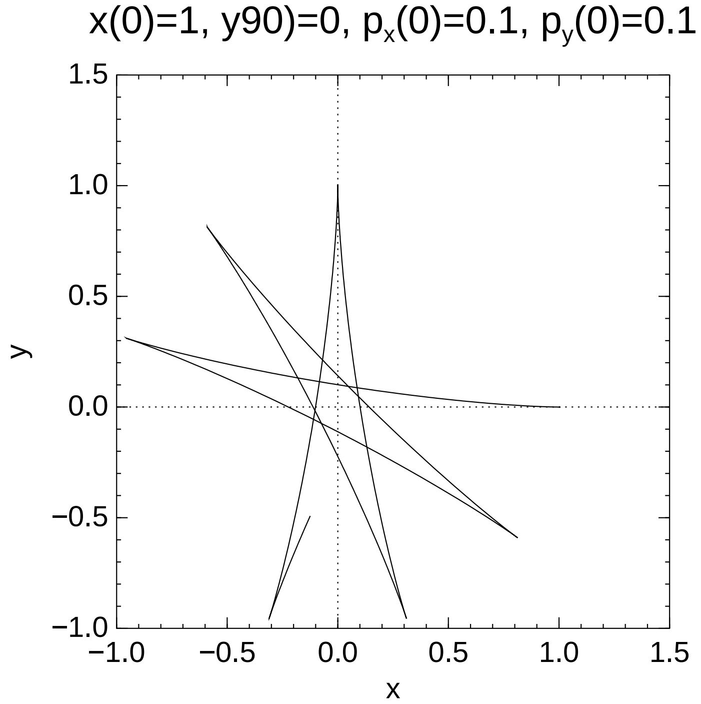
$$\text{let} \quad \xi = \Xi\, e^{-i\omega t}, \quad \eta = \Upsilon\, e^{-i\omega t}, \quad \text{then} \quad \dot{\Xi} = \frac{1}{m}\Upsilon, \qquad \dot{\Upsilon} = -m\,\Omega^2\,\Xi$$

and let $m = 1$ so it is not floating around. Now re-use problem **18**, let

$$\zeta_\pm = \Upsilon \pm i\Omega\,\Xi, \qquad \dot{\zeta}_\pm = \dot{\Upsilon} \pm i\dot{\Xi} = -\Omega^2\,\Xi \pm i\Omega\,\Upsilon, \qquad \dot{\zeta}_\pm = \pm i\Omega\,\zeta_\pm, \qquad \zeta_\pm = e^{\pm i\Omega t}\,\zeta_\pm(0)$$

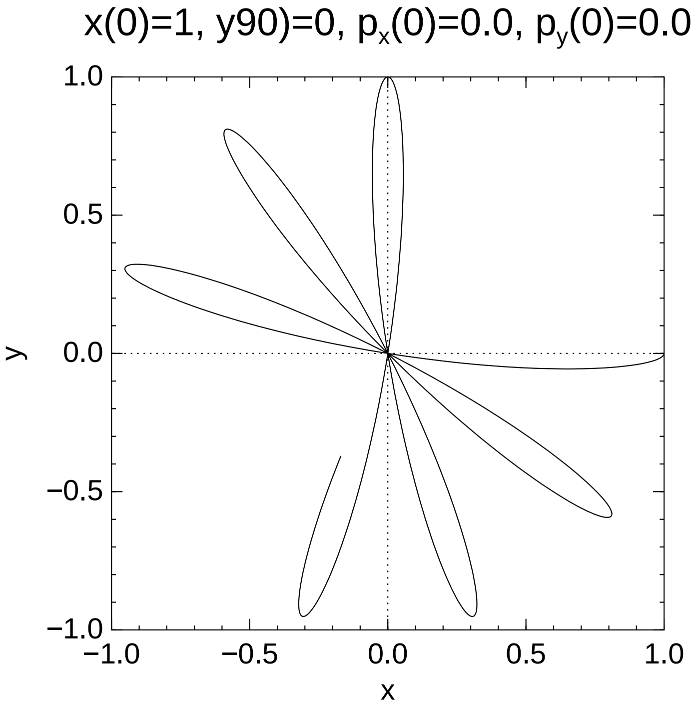
from which we can recover

$$\xi = e^{-i\omega t}\,\frac{1}{i\Omega}\Big(\zeta_+ - \zeta_-\Big) = e^{i\omega t}\,\frac{1}{i\Omega}\Big(\zeta_+(0)\,e^{i\Omega t} - \zeta_-(0)\,e^{-i\Omega t}\Big) = e^{i\omega t}\Big(A\,e^{i\Omega t} + B\,e^{-i\Omega t}\Big)$$



Of course you could solve these by more traditional means, but our purpose here is to learn to fully exploit the first-ordered-ness of these equations of motion. For example, lets do it with **ode**

```
O=1.0
o=0.1
px'=o*py-O*O*x
py'=-o*px-O*O*y
x'=px+o*y
y'=py-o*x
x=1.0
y=0.0
px=0.0
py=0.0
print x,y
step 0, 20
```


x(0)=1, y90)=0, p$_x$(0)=0.0, p$_y$(0)=0.0

We do this for two sets of initial data just so that you can see how the Coriolis terms affect the motion. They cause **precession**, the "orbit" of the oscillator does not re-trace its own path. A pendulum swinging at any point on the earth's surface will exhibit this precession, with a period of 24 hours, which I offer up as proof the it is the earth that rotates, and not the rest of the universe. The rotation of the earth creates the Coriolis force that causes the precession. This is called **Foucault's pendulum**.

This problem is very important because of its concrete physical connections, and because (depending on how you work, and how deeply you apply yourself to your work) you can solve it in many ways that appear almost novel. If you study Eqs.1 quite carefully, you can see that they have a block structure; both $\dot{p}_x$ and $\dot{y}$ depend **only** on p_y and x, and vice versa. Therefore lets invent! (Let $m = 1$)

$$\begin{pmatrix} \dot{x} \\ \dot{p}_y \\ \dot{p}_x \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & \omega \\ 0 & 0 & -\omega & -\Omega^2 \\ -\Omega^2 & \omega & 0 & 0 \\ -\omega & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ p_y \\ p_x \\ y \end{pmatrix}, \quad \text{Call this} \quad \dot{\boldsymbol{\xi}} = \mathbf{M} \cdot \boldsymbol{\xi}$$

Start playing around with this, and you discover that

$$\mathbf{M} \cdot \mathbf{M} = -(\omega^2 + \Omega^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -(\omega^2 + \Omega^2)\, 1$$

From this we can deduce that

$$\Big(\mathbf{M}\Big)^{2n} = (-1)^n(\omega^2 + \Omega^2), \qquad \Big(\mathbf{M}\Big)^{2n+1} = (-1)^n(\omega^2 + \Omega^2), \mathbf{M}$$

and insert this into the Taylor series for $\boldsymbol{\xi}(t)$

$$\boldsymbol{\xi}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{d^n}{dt^n} \boldsymbol{\xi}(t)\Big)_{t=0} t^n = \sum_{n\,odd}^{\infty} \frac{1}{n!} \Big(\frac{d^n}{dt^n} \boldsymbol{\xi}(t)\Big)_{t=0} t^n + \sum_{n\,even}^{\infty} \frac{1}{n!} \Big(\frac{d^n}{dt^n} \boldsymbol{\xi}(t)\Big)_{t=0} t^n$$

just like you did in our simple series solutions from the first assignment. You obtain

$$\boldsymbol{\xi}(t) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \Big(\mathbf{M}\Big)^{2n+1} \boldsymbol{\xi}(0)\, t^{2n+1} + \sum_{n=0}^{\infty} \frac{1}{(2n)!} \Big(\mathbf{M}\Big)^{2n} \boldsymbol{\xi}(0)\, t^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\Big(-(\omega^2+\Omega^2)\Big)^n \mathbf{M}\cdot\boldsymbol{\xi}(0)\, t^{2n+1} + \sum_{n=0}^{\infty}\frac{1}{(2n)!}\Big(-(\omega^2+\Omega^2)\Big)^n \boldsymbol{\xi}(0)\, t^{2n}$$

$$= \frac{\sin(\sqrt{\omega^2+\Omega^2}\, t)}{\sqrt{\omega^2+\Omega^2}}\, \mathbf{M}\cdot\boldsymbol{\xi}(0) + \cos(\sqrt{\omega^2+\Omega^2}\, t)\, \boldsymbol{\xi}(0)$$

which is a pretty spiffy route to a solution that is very similar to group-theoretical methods that we will soon apply to quantum systems.

20.

This is an important example. Writing it out explicitly

$$L = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\frac{B_z}{2}y\dot{x} + q\frac{B_z}{2}x\dot{y}$$

this looks suspiciously similar to the previous problem but without the centripetal terms.

$$p_x = m\dot{x} - q\frac{B_z}{2}y, \quad p_y = m\dot{y} + q\frac{B_z}{2}x, \quad p_z = m\dot{z}, \qquad \dot{x} = \frac{p_x + \frac{qB_z}{2}y}{m}, \quad \dot{y} = \frac{p_y - \frac{qB_z}{2}x}{m}, \quad \dot{z} = \frac{p_z}{m}$$

The Lagrangian equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \qquad \frac{d}{dt}(m\dot{x} - q\frac{B_z}{2}y) = q\frac{B_z}{2}\dot{y}, \qquad m\ddot{x} = qB_z\dot{y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}, \qquad \frac{d}{dt}(m\dot{y} + q\frac{B_z}{2}x) = -q\frac{B_z}{2}\dot{x}, \qquad m\ddot{y} = -qB_z\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z}, \qquad \frac{d}{dt}(m\dot{z}) = 0$$

which are precisely $m\dot{\mathbf{v}} = q\mathbf{v}\times\mathbf{B}$ for $\mathbf{B} = B_z\mathbf{k}$.

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L = \frac{1}{2m}\Big((p_x + \frac{qB_z}{2}y)^2 + (p_y - \frac{qB_z}{2}x)^2\Big)$$

In the most general case, with an arbitrary vector potential $\mathbf{A}$ we would get

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})\cdot(\mathbf{p} - q\mathbf{A})$$

21.Henon-Heiles is important because it behaves chaotically. If you are interested in chaos, see Appendix C, and the books by Drazin and Steeb in the bibliography.

$$p_1 = \dot{x}_1, \quad p_2 = \dot{x}_2, \qquad H = \tfrac{1}{2}(p_1^2+p_2^2) + \tfrac{1}{3}x_1^3 - x_1x_2^2$$

$$\dot{x}_1 = p_1, \quad \dot{x}_2 = p_2, \qquad \dot{p}_1 = -(x_1^2 - x_2^2), \quad \dot{p}_2 = 2x_1x_2$$

22.

This is simple, just show that all seven requirements are met;

Closure; $(a+ib)+(c+id) = (a+c)+i(b+d) \in \{x+iy\,|\,x,y\in\mathbb{R}\}$

Unique identity; $(a+ib)+(0+i0) = a+ib, \qquad 0+i0 \in \{x+iy\,|\,x,y\in\mathbb{R}\}$

Unique inverses; $(a+ib)+(-a+i(-b)) = 0+i0, \qquad \text{since} \quad a\in\mathbb{R} \implies -a\in\mathbb{R}$

Associativity follows from associativity of numerical addition in the Reals.

Monoid/ring properties; $\lambda(a+ib) = \lambda a + i(\lambda b), \quad \text{since} \quad \lambda, a, b \in\mathbb{R} \implies \lambda a, \lambda b\in\mathbb{R}$.

$\mu(\lambda(a+ib)) = \mu(\lambda a + i(\lambda b)) = (\mu\lambda a) + i(\mu\lambda b) \in \{x + iy \,|\, x, y \in \mathbb{R}\}$, and so forth.

23. Using this bilinear form you can show that $Tr(\mathbf{e}_i \cdot \mathbf{f}_j) = \delta_{ij}$, where $\mathbf{f}_i = \mathbf{e}_i^T$ (transpose) for example

$$\mathbf{e}_1 \cdot \mathbf{f}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Tr(\mathbf{e}_1 \cdot \mathbf{f}_2) = 0$$

$$\mathbf{e}_1 \cdot \mathbf{f}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Tr(\mathbf{e}_1 \cdot \mathbf{f}_1) = 1$$

and so one, establishing that $\mathbf{e}_i^* = \mathbf{f}_i = \mathbf{e}_i^T$.

24. Simply use the equations of motion.

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = -Aq_1 - q_1^3 - q_1 q_2^2, \quad \dot{p}_2 = -Aq_2 - q_2^3 - q_2 q_1^2$$

so

$$\dot{I}_1 = \dot{q}_1 p_2 - \dot{q}_2 p_1 + q_1 \dot{p}_2 - q_2 \dot{p}_1 = p_1 p_2 - p_2 p_1 + q_1\Big(-Aq_2 - q_2^3 - q_2 q_1^2\Big) - q_2\Big(-Aq_1 - q_1^3 - q_1 q_2^2\Big) = 0$$

and so forth.

25.

$$\dot{q} = 2p^3, \quad \dot{p} = -2q^3$$

$$\dot{H} = 2p^3\dot{p} + 2q^3\dot{q} = 2p^3(-2q^3) + 2q^3(2p^3) = 0, \qquad p = \sqrt[4]{2C - q^4}$$

so

$$2\,dt = \frac{dq}{(2C - q^4)^{\frac{3}{4}}}, \qquad 2(t - t_0) = \int \frac{dq}{(2C - q^4)^{\frac{3}{4}}}$$

This is a nasty one! I'm just kidding, its not as icky as it looks, but the answer is not a function that you are intimately familiar with (yet). **Even if you can't integrate this, you can always get the period** of the motion. One period is the time it takes q to go from $-q_{max} = -\sqrt[4]{2C}$ to $q_{max} = \sqrt[4]{2C}$ back to $-q_{max}$, which is 4 times what it takes to go from 0 to q_{max};

$$2T = 4\int_0^{\sqrt[4]{2C}} \frac{dq}{(2C - q^4)^{\frac{3}{4}}}, \qquad \text{Let} \quad q = \sqrt[4]{2C}x; \qquad 2T = 4\frac{1}{\sqrt{2C}}\int_0^1 \frac{dx}{(1 - x^4)^{\frac{3}{4}}}$$

This last integral is not elementary, but it is not too difficult, **Maxima** gives it as $\frac{1}{4}\beta(\frac{1}{4}, \frac{1}{4})$ where $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. See Lebedev, *Special Functions*, p.13.

26, **27**. These "problems" are simply activities designed to introduce you to **ode**. Some **ode** scripts are available for download from the 441 website under **computer programs**. **Ode** provides you with a way of solving Hamiltonian equations of motion by computer without the need to learn any programming.

28,**29**. These "problems" are activities designed to introduce you to Runge Kutta extrapolation, which David will tell you about in **303**. Some sample programs in C, Vpython and java are available for download from the 441 website under **computer programs**. I don't remember Fortran well enough to provide you with Fortran examples, perhaps you could write some as joint 303/441 activities?

```
C     Phase space program for quartic oscillator
C     export LD_LIBRARY_PATH=$LD_LIBRARY_PATH:/usr/local/fortran/lib
C                                            :/usr/local/fortran/lib64
C     gfortran osc.f -o osc.exe

      real phase
      real x, v, t, x0, v0,a,b,c,d
      real dt, dev, dev1, q, s
      dt = 0.003
      x0 = 1.0
      v0 = 0.0
      s=0.15

C     Simple RK integration
      do 100 k=1, 5000
         k1=h*p;
         l1=-2.0*h*q*q*q;
         k2=h*(p+0.5*l1);
         l2=-2.0*h*(q+0.5*k1)*(q+0.5*k1)*(q+0.5*k1);
         k3=h*(p+l2);
         l3=-2.0*h*(q+k2)*(q+k2)*(q+k2);
         q=q+(k1+4.0*k2+k3)/6.0;
         p=p+(l1+4.0*l2+l3)/6.0;
         write(6,*) q,p
 100     continue
      stop
      end
```

30. $\mathfrak{C}(Q_{1,1})$ has two generators

$$\mathbf{e}_1\#\mathbf{e}_1+\mathbf{e}_1\#\mathbf{e}_1 = 2Q_{1,1}(\mathbf{e}_1,\mathbf{e}_1) = 2, \qquad \mathbf{e}_2\#\mathbf{e}_2+\mathbf{e}_2\#\mathbf{e}_2 = 2Q_{1,1}(\mathbf{e}_2,\mathbf{e}_2) = -2, \qquad \mathbf{e}_1\#\mathbf{e}_2+\mathbf{e}_2\#\mathbf{e}_1 = 2Q_{1,1}(\mathbf{e}_1,\mathbf{e}_2) = 0$$

or

$$\mathbf{e}_1\#\mathbf{e}_1 = 1, \qquad \mathbf{e}_2\#\mathbf{e}_2 = -1, \qquad \mathbf{e}_1\#\mathbf{e}_2 + \mathbf{e}_2\#\mathbf{e}_1 = 0$$

The last of which cannot be obeyed by real or complex numbers, so we represent these objects by matrices of the smallest size possible, and represent # as matrix multiplication

$$\rho(\mathbf{e}_1)\rho(\mathbf{e}_1) = 1, \qquad \rho(\mathbf{e}_2)\rho(\mathbf{e}_2) = -1, \qquad \rho(\mathbf{e}_1)\rho(\mathbf{e}_2) + \rho(\mathbf{e}_2)\rho(\mathbf{e}_1) = 0$$

which is the sense in which the "equality" between $\mathbf{e}_i$ and matrices in the posing of the problem is to be understood. We find that

$$\rho(\mathbf{e}_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(\mathbf{e}_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(\mathbf{e}_1\#\mathbf{e}_2) = \rho(\mathbf{e}_1)\rho(\mathbf{e}_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

together with $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ complete the entire Clifford algebra.

Here is an important calculation;

$$(\mathbf{e}_1\#\mathbf{e}_2)\#(\mathbf{e}_1\#\mathbf{e}_2) = \mathbf{e}_1\#(\mathbf{e}_2\#\mathbf{e}_1)\#\mathbf{e}_2 = \mathbf{e}_1\#(-\mathbf{e}_1\#\mathbf{e}_2)\#\mathbf{e}_2 = -(\mathbf{e}_1\#\mathbf{e}_1)\#(\mathbf{e}_2\#\mathbf{e}_2) = 1$$

all by associativity of #.
The set of real, 2×2 matrices $\mathbb{R}(2)$ is

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

can be easily shown to be a vector space with basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and we can construct an isomorphism $\phi : \mathfrak{C}(Q_{1,1}) \to \mathbb{R}(2)$;

$$\phi(\tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_2)) = \mathbf{v}_1, \qquad \phi(\tfrac{1}{2}(1 + \mathbf{e}_1 \# \mathbf{e}_2)) = \mathbf{v}_4, \quad \phi(\tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)) = \mathbf{v}_2, \quad \phi(\tfrac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2)) = \mathbf{v}_3$$

which is an isomorphism of vector spaces (it honors all algebraic relations within both vector spaces). Since we have not defined a bilinear product in $\mathbb{R}(2)$, yet this is not an isomorphism of algebras. Lets define a bilinear mapping in $\mathbb{R}(2)$, $V \otimes V \to V$, simple matrix multiplication. Then

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = \mathbf{v}_1, \qquad \mathbf{v}_1 \cdot \mathbf{v}_4 = \mathbf{v}_4 \cdot \mathbf{v}_1 = 0, \qquad \mathbf{v}_4 \cdot \mathbf{v}_4 = \mathbf{v}_4$$

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = \mathbf{v}_3 \cdot \mathbf{3} = 0, \qquad \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_1, \qquad \mathbf{v}_3 \cdot \mathbf{v}_2 = \mathbf{v}_4, \quad \mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2$$

and so forth. The map ϕ is now easily shown to be an isomorphism of algebras, for example

$$\phi(\tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_1) \# \tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_1)) = \phi(\frac{1}{4}(1 - 2\mathbf{e}_1 \# \mathbf{e}_1 + \mathbf{e}_1 \# (\mathbf{e}_1 \# \mathbf{e}_1) \# \mathbf{e}_1) = \phi(\tfrac{1}{2}(1 - \mathbf{e}_1)) = \mathbf{v}_1$$

so that

$$\phi(\tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_1) \# \tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_1)) = \phi(\tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_1) \cdot \phi(\tfrac{1}{2}(1 - \mathbf{e}_1 \# \mathbf{e}_1)) = \mathbf{v}_1 \cdot \mathbf{v}_1$$

and so
31. $\mathfrak{C}(Q_{0,2})$ has two generators

$$\mathbf{e}_1 \# \mathbf{e}_1 + \mathbf{e}_1 \# \mathbf{e}_1 = 2Q_{0,2}(\mathbf{e}_1, \mathbf{e}_1) = -2, \qquad \mathbf{e}_2 \# \mathbf{e}_2 + \mathbf{e}_2 \# \mathbf{e}_2 = 2Q_{0,2}(\mathbf{e}_2, \mathbf{e}_2) = -2$$

$$\mathbf{e}_1 \# \mathbf{e}_2 + \mathbf{e}_2 \# \mathbf{e}_1 = 2Q_{0,2}(\mathbf{e}_1, \mathbf{e}_2) = 0$$

or

$$\mathbf{e}_1 \# \mathbf{e}_1 = -1, \qquad \mathbf{e}_2 \# \mathbf{e}_2 = -1, \qquad \mathbf{e}_1 \# \mathbf{e}_2 + \mathbf{e}_2 \# \mathbf{e}_1 = 0$$

The last of which cannot be obeyed by real or complex numbers, so we represent these objects by matrices of the smallest size possible, and represent # as matrix multiplication. **Here is an important calculation**;

$$(\mathbf{e}_1 \# \mathbf{e}_2) \# (\mathbf{e}_1 \# \mathbf{e}_2) = \mathbf{e}_1 \# (\mathbf{e}_2 \# \mathbf{e}_1) \# \mathbf{e}_2 = \mathbf{e}_1 \# (-\mathbf{e}_1 \# \mathbf{e}_2) \# \mathbf{e}_2 = -(\mathbf{e}_1 \# \mathbf{e}_1) \# (\mathbf{e}_2 \# \mathbf{e}_2) = -1$$

and so if we call $\mathbf{i} = \mathbf{e}_1$, $\mathbf{j} = \mathbf{e}_2$, $\mathbf{k} = \mathbf{e}_1 \# \mathbf{e}_2$, we have three objects that obey

$$\mathbf{i} \# \mathbf{i} = \mathbf{j} \# \mathbf{j} = \mathbf{k} \# \mathbf{k} = -1$$

(which account for the Quaternions being called hypercomplex numbers), and

$$\begin{aligned}
\mathbf{i}\#\mathbf{j} &= \mathbf{e}_1\#\mathbf{e}_2 = \mathbf{k}, \qquad \mathbf{j}\#\mathbf{i} = \mathbf{e}_2\#\mathbf{e}_1 = -\mathbf{e}_1\#\mathbf{e}_2 = -\mathbf{k} \\
\mathbf{i}\#\mathbf{k} &= \mathbf{e}_1\#(\mathbf{e}_1\#\mathbf{e}_2) = (\mathbf{e}_1\#\mathbf{e}_1)\#\mathbf{e}_2) = -\mathbf{e}_2 = -\mathbf{j} \\
\mathbf{k}\#\mathbf{i} &= (\mathbf{e}_1\#\mathbf{e}_2)\#\mathbf{e}_1 = -(\mathbf{e}_2\#\mathbf{e}_1)\#\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{j} \\
\mathbf{j}\#\mathbf{k} &= \mathbf{e}_2\#(\mathbf{e}_1\#\mathbf{e}_2) = (\mathbf{e}_2\#\mathbf{e}_1)\#\mathbf{e}_2 = -(\mathbf{e}_1\#\mathbf{e}_2)\#\mathbf{e}_2 = \mathbf{e}_1 = \mathbf{i} \\
\mathbf{k}\#\mathbf{j} &= (\mathbf{e}_1\#\mathbf{e}_2)\#\mathbf{e}_2 = \mathbf{e}_1\#(\mathbf{e}_2\#\mathbf{e}_2) = -\mathbf{e}_1 = -\mathbf{i} \qquad (2)
\end{aligned}$$

It is a simple matter of matrix arithetic to show that the representation (injection into a set of matrices)

$$\phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi(\mathbf{i}) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \phi(\mathbf{j}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \phi(\mathbf{k}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

honors all nine of these relations. We sometimes write these as **anticommutators**

$$\{\mathbf{i},\mathbf{j}\} \equiv \mathbf{i}\#\mathbf{j} + \mathbf{j}\#\mathbf{i} = 0, \quad \{\mathbf{i},\mathbf{k}\} = 0, \quad \{\mathbf{j},\mathbf{k}\} = 0, \quad \{\mathbf{i},\mathbf{i}\} = -2, \cdots$$

or calling $\mathbf{i} = \sigma_1$, $\mathbf{j} = \sigma_2$, $\mathbf{k} = \sigma_3$

$$\{\sigma_i, \sigma_j\} = -2\delta_{ij}$$

What compounds the excitement is that according to Eq.2 these objects obey a Lie algebra

$$[\mathbf{i},\mathbf{j}] \equiv \mathbf{i}\#\mathbf{j} - \mathbf{j}\#\mathbf{i} = 2\mathbf{k}, \qquad [\mathbf{k},\mathbf{i}] = 2\mathbf{j}, \qquad [\mathbf{j},\mathbf{k}] = 2\mathbf{i}$$

or

$$[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\,\sigma_k$$

You will see this Lie algebra every day from now on! Where?
1. The set of tangent vector fields to a sphere obey these relations,
2. so do the quantum orbital angular momentum operators,
3. so do the quantum spin operators.

32. Just calculate using the Clifford relations given (and associativity of #)

$$x_3x_1 = x_3\#x_1 = (\mathbf{e}^1\#\mathbf{e}^2)\#(\mathbf{e}^2\#\mathbf{e}^3) = \mathbf{e}^1\#(\mathbf{e}^2\#\mathbf{e}^2)\#\mathbf{e}^3 = \mathbf{e}^1\#\mathbf{e}^3 = -x_2$$

$$x_1\#x_3 = (\mathbf{e}^2\#\mathbf{e}^3)\#(\mathbf{e}^1\#\mathbf{e}^2) = -(\mathbf{e}^3\#\mathbf{e}^2)\#(-\mathbf{e}^2\#\mathbf{e}^1) = \mathbf{e}^1\#\mathbf{e}^3 = x_2$$

Subtract

$$x_1\#x_3 - x_3\#x_1 \equiv [x_1,\, x_3] = 2x_2$$

$$x_2\#x_1 = (\mathbf{e}^3\#\mathbf{e}^1)\#(\mathbf{e}^2\#\mathbf{e}^3) = (-\mathbf{e}^1\#\mathbf{e}^3)\#(-\mathbf{e}^3\#\mathbf{e}^2) = \mathbf{e}^1\#\mathbf{e}^2 = x_3$$

$$x_1\#x_2 = (\mathbf{e}^2\#\mathbf{e}^3)\#(\mathbf{e}^3\#\mathbf{e}^1) = \mathbf{e}^2\#(\mathbf{e}^3\#\mathbf{e}^3)\#\mathbf{e}^1 = \mathbf{e}^2\#\mathbf{e}^1 = -x_3$$

Subtract

$$x_1\#x_2 - x_2\#x_1 \equiv [x_1,\, x_2] = -2x_3$$

Good Glaven, its the same as the Lie algebra of the Quaternions! Do you think that this could mean something?

33. First of all we need a "sum of squares" metric, and associated tangent vectors

$$g = dr\otimes dr + r^2\, d\phi\otimes d\phi + dz\otimes dz = \mathbf{e}^1\otimes\mathbf{e}^1 + \mathbf{e}^2\otimes\mathbf{e}^2 + \mathbf{e}^3\otimes\mathbf{e}^3$$

$$\mathbf{e}^1 = dr, \qquad \mathbf{e}^2 = r\,d\phi, \qquad \mathbf{e}^3 = dz, \qquad \mathbf{E}_1 = \frac{\partial}{\partial r}, \qquad \mathbf{E}_2 = \frac{1}{r}\frac{\partial}{\partial \phi} \qquad \mathbf{E}_3 = \frac{\partial}{\partial z}$$

and remember that we associate $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ with unit vectors (co-vectors) $\{\hat{r}, \hat{\phi}, \hat{z}\}$.
Let $\boldsymbol{\omega}_1 = a\,\mathbf{e}^1 + b\,\mathbf{e}^2 + c\,\mathbf{e}^3$ be a one-form. We work out the full set of hodge-star relations first

$$*\mathbf{e}^1 = \mathbf{e}^2 \wedge \mathbf{e}^3, \qquad *\mathbf{e}^2 = -\mathbf{e}^1 \wedge \mathbf{e}^3, \qquad *\mathbf{e}^3 = \mathbf{e}^1 \wedge \mathbf{e}^2$$

$$*\mathbf{e}^1 \wedge \mathbf{e}^2 = \mathbf{e}^3, \qquad *\mathbf{e}^2 \wedge \mathbf{e}^3 = \mathbf{e}^1, \qquad *\mathbf{e}^1 \wedge \mathbf{e}^3 = -\mathbf{e}^2$$

$$*1 = \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3, \qquad *\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 = 1$$

and

$$d\mathbf{e}^1 = 0, \qquad d\mathbf{e}^2 = dr \wedge d\phi = \frac{1}{r}\mathbf{e}^1 \wedge \mathbf{e}^2, \qquad d\mathbf{e}^3 = 0, \quad [E_1, E_2] = -\frac{1}{r}E_2, \quad [E_1, E_3] = [E_2, E_3] = 0$$

$$d(\mathbf{e}^1 \wedge \mathbf{e}^2) = (d\mathbf{e}^1) \wedge \mathbf{e}^2 - \mathbf{e}^1 \wedge (d\mathbf{e}^2) = 0, \qquad d(\mathbf{e}^1 \wedge \mathbf{e}^3) = 0, \qquad d(\mathbf{e}^2 \wedge \mathbf{e}^3) = \frac{1}{r}\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3$$

So here it goes;

$$\begin{aligned}
d\boldsymbol{\omega}_1 &= da \wedge \mathbf{e}^1 + db \wedge \mathbf{e}^2 + dc \wedge \mathbf{e}^3 + b\frac{1}{r}\mathbf{e}^1 \wedge \mathbf{e}^2 \\
&= \Big(E_2(a)\,\mathbf{e}^2 + E_3(a)\mathbf{e}^3\Big) \wedge \mathbf{e}^1 + \Big(E_1(b)\,\mathbf{e}^1 + E_3(b)\mathbf{e}^3\Big) \wedge \mathbf{e}^2 + \Big(E_1(c)\,\mathbf{e}^1 + E_2(c)\mathbf{e}^2\Big) \wedge \mathbf{e}^3 + b\frac{1}{r}\mathbf{e}^1 \wedge \mathbf{e}^2 \\
&= \Big(E_1(b) - E_2(a) + \frac{b}{r}\Big)\mathbf{e}^1 \wedge \mathbf{e}^2 + \Big(E_2(c) - E_3(b)\Big)\mathbf{e}^2 \wedge \mathbf{e}^3 + \Big(E_3(a) - E_1(c)\Big)\mathbf{e}^3 \wedge \mathbf{e}^1 \\
*d\boldsymbol{\omega}_1 &= \Big(E_1(b) - E_2(a) + \frac{b}{r}\Big)\mathbf{e}^3 + \Big(E_2(c) - E_3(b)\Big)\mathbf{e}^1 + \Big(E_3(a) - E_1(c)\Big)\mathbf{e}^2 \\
d * d\boldsymbol{\omega}_1 &= E_1\Big(E_1(b) - E_2(a) + \frac{b}{r}\Big)\mathbf{e}^1 \wedge \mathbf{e}^3 + E_2\Big(E_1(b) - E_2(a) + \frac{b}{r}\Big)\mathbf{e}^2 \wedge \mathbf{e}^3 \\
&+ E_2\Big(E_2(c) - E_3(b)\Big)\mathbf{e}^2 \wedge \mathbf{e}^1 + E_3\Big(E_2(c) - E_3(b)\Big)\mathbf{e}^3 \wedge \mathbf{e}^1 \\
&+ E_1\Big(E_3(a) - E_1(c)\Big)\mathbf{e}^1 \wedge \mathbf{e}^2 + E_3\Big(E_3(a) - E_1(c)\Big)\mathbf{e}^3 \wedge \mathbf{e}^2 + \frac{1}{r}\Big(E_3(a) - E_1(c)\Big)\mathbf{e}^1 \wedge \mathbf{e}^2 \\
*d * d\boldsymbol{\omega}_1 &= -E_1\Big(E_1(b) - E_2(a) + \frac{b}{r}\Big)\mathbf{e}^2 + E_2\Big(E_1(b) - E_2(a) + \frac{b}{r}\Big)\mathbf{e}^1 \\
&- E_2\Big(E_2(c) - E_3(b)\Big)\mathbf{e}^3 + E_3\Big(E_2(c) - E_3(b)\Big)\mathbf{e}^2 \\
&+ E_1\Big(E_3(a) - E_1(c)\Big)\mathbf{e}^3 - E_3\Big(E_3(a) - E_1(c)\Big)\mathbf{e}^1 + \frac{1}{r}\Big(E_3(a) - E_1(c)\Big)\mathbf{e}^3 \\
&= \delta(d\boldsymbol{\omega}_1) \qquad (3)
\end{aligned}$$

(see the definition of δ following Eq.4.50), since $d\boldsymbol{\omega}_1$ is a two-form ($p = 2$, $n = 3$).

$$\begin{aligned}
*\boldsymbol{\omega}_1 &= a\,\mathbf{e}^2\wedge\mathbf{e}^3 - b\,\mathbf{e}^1\wedge\mathbf{e}^3 + c\,\mathbf{e}^1\wedge\mathbf{e}^2\\
d*\boldsymbol{\omega}_1 &= da\,\mathbf{e}^2\wedge\mathbf{e}^3 - db\,\mathbf{e}^1\wedge\mathbf{e}^3 + dc\,\mathbf{e}^1\wedge\mathbf{e}^2 + \frac{a}{r}\,\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3\\
&= \Big(E_1(a)+\frac{a}{r}\Big)\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3 + \Big(E_2(b)\Big)\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3 + \Big(E_3(c)\Big)\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3\\
&= \Big(E_1(a)+E_2(b)+E_3(c)+\frac{a}{r}\Big)\mathbf{e}^1\wedge\mathbf{e}^2\wedge\mathbf{e}^3\\
d\boldsymbol{\omega}_1 &= \Big(E_1(a)+E_2(b)+E_3(c)+\frac{a}{r}\Big)\\
d*d*\boldsymbol{\omega}_1 &= E_1\Big(E_1(a)+E_2(b)+E_3(c)+\frac{a}{r}\Big)\,\mathbf{e}^1 + E_2\Big(E_1(a)+E_2(b)+E_3(c)+\frac{a}{r}\Big)\,\mathbf{e}^2\\
&+\ E_3\Big(E_1(a)+E_2(b)+E_3(c)+\frac{a}{r}\Big)\,\mathbf{e}^3\\
&= -d(\delta\boldsymbol{\omega}_1)
\end{aligned}\tag{4}$$

because $\delta\boldsymbol{\omega}_1 = (-1)^{3(1+1)-1} * d * \boldsymbol{\omega}_1$. In other words $\nabla^2\mathbf{v} = \boldsymbol{\nabla}\Big(\boldsymbol{\nabla}\cdot\mathbf{v}\Big) - \boldsymbol{\nabla}\times\Big(\boldsymbol{\nabla}\times\mathbf{v}\Big)$.

$$\begin{aligned}
(d+\delta)^2\boldsymbol{\omega}_1 &= -\Big((E_1^2+E_2^2+E_3^2)a + E_2(\frac{b}{r}) - E_1(\frac{a}{r}) - [E_1,E_2]b\Big)\,\mathbf{e}^1\\
&+\ -\Big((E_1^2+E_2^2+E_3^2)b - E_2(\frac{a}{r}) - E_1(\frac{b}{r}) + [E_1,E_2]a\Big)\,\mathbf{e}^2\\
&+\ -\Big((E_1^2+E_2^2+E_3^2)c - \frac{1}{r}E_1(c)\Big)\,\mathbf{e}^3
\end{aligned}\tag{5}$$

In physics literature (old literature) you will see

$$\boldsymbol{\omega}_1 = \omega_r\,\hat{r} + \omega_\phi\,\hat{\phi} + \omega_z\,\hat{z}$$

and

$$\begin{aligned}
\nabla^2\boldsymbol{\omega}_1 &= -\Big((E_1^2+E_2^2+E_3^2)\omega_r + E_2(\frac{\omega_\phi}{r}) - E_1(\frac{\omega_r}{r}) - [E_1,E_2]\omega_\phi\Big)\,\hat{r}\\
&+\ -\Big((E_1^2+E_2^2+E_3^2)\omega_\phi - E_2(\frac{\omega_r}{r}) - E_1(\frac{\omega_\phi}{r}) + [E_1,E_2]\omega_r\Big)\,\hat{\phi}\\
&+\ -\Big((E_1^2+E_2^2+E_3^2)\omega_z - \frac{1}{r}E_1(\omega_z)\Big)\,\hat{z}\\
&= -\Big((\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2})\omega_r + \frac{2}{r}\frac{\partial}{\partial\phi}(\frac{\omega_\phi}{r}) - \frac{\partial}{\partial r}(\frac{\omega_r}{r})\Big)\,\hat{r}\\
&+\ -\Big((\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2})\omega_\phi - \frac{2}{r}\frac{\partial}{\partial\phi}(\frac{\omega_r}{r}) - \frac{\partial}{\partial r}(\frac{\omega_\phi}{r})\Big)\,\hat{\phi}\\
&+\ -\Big((\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2})\omega_z - \frac{1}{r}\frac{\partial}{\partial r}(\omega_z)\Big)\,\hat{z}
\end{aligned}\tag{6}$$

The labour for the spherical case is nearly the same, but about 25% longer. You can check your results against http://mathworld.wolfram.com/VectorLaplacian.html