Physics 441 Homework V solutions

47. Begin with $F = S = W(q, E) - Et$, and $p = \frac{\partial S}{\partial q}$ to get

$$\frac{1}{2m}\Big(\frac{\partial S}{\partial q}\Big)^2 + mgq - E = 0, \qquad S = \int \sqrt{2m(E - mgq)}\, dq - Et$$

$$\beta = \frac{\partial S}{\partial E} = \tfrac{1}{2}\int \sqrt{\frac{2m}{(E - mgq)}}\, dq - t = \frac{1}{mg}\sqrt{2m(E - mgq)} - t$$

and isolate q

$$q = \frac{E}{mg} - \tfrac{1}{2}g(t + \beta)^2 = h - \tfrac{1}{2}g(t + \beta)^2$$

This is a heavy-handed way to solve a simple problem, but the beauty of the HJ approach is that it is no more difficult to solve a complex problem than a simple one by this approach.

48.

This one can be done by straight-forward integration, and is in fact a special case. **There are no "orbits" unless the mass can bounce** off of the ground, and this interaction is not built into the Hamiltonian. Therefore integration by Cauchy's theorem is not possible because the dynamical description of orbits is incomplete in some sense, the ground provides the necessary restoring force.

$$S = \oint_C \sqrt{2m(E - mgq)}\, dq = \oint_C \sqrt{2m(mgH - mgq)}\, dq = 2\sqrt{2m^2 g}\int_0^H \sqrt{(H - q)}\, dq = \frac{4}{3}\sqrt{2m^2 g H^3} = nh$$

which implies that a bouncing ball can't have any energy, but only energies

$$mgH = mg\sqrt[3]{\frac{9}{16}\frac{n^2h^2}{2m^2 g}}$$

49. The most intelligent approach to Bohr-Sommerfeld-Wilson quantization is to always do the integrals using the Cauchy theorem

$$\oint_c \frac{dq}{q} = 2\pi i, \qquad \oint_c q^n\, dq = 0, \quad n \neq -1$$

$$H = E = \frac{1}{2m}\Big(p_r^2 + \frac{p_\phi^2}{r^2}\Big) + \tfrac{1}{2}m\omega^2 r^2$$

We obtain canonical momenta

$$p_\phi = mr^2\dot{\phi}, \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0, \qquad p_r = m\dot{r}$$

therefore p_ϕ is a constant, so

$$\oint p_\phi d\phi = p_\phi \oint d\phi = 2\pi p_\phi = \ell h, \qquad p_\phi = \ell\frac{h}{2\pi} = \ell\hbar$$

and

$$E = \frac{1}{2m}p_r^2 + \frac{\ell^2\hbar^2}{2mr^2} + \tfrac{1}{2}m\omega^2 r^2, \qquad \oint p_r\, dr = \oint \sqrt{2m(E - \frac{\ell^2\hbar^2}{2mr^2} - \tfrac{1}{2}m\omega^2 r^2)}\, dr = nh$$

so we do this integral the smart way, expanding it in series near the non-analytic points

$$\begin{aligned} nh &= \oint \sqrt{2m\Big(-\frac{\ell^2\hbar^2}{2mr^2}\Big)}\,dr + \oint \sqrt{(-m^2\omega^2r^2)(1-\frac{2E}{m\omega^2r^2})}\,dr + \cdots \\ &= 2\pi i\sqrt{-1}\,\ell\hbar - im\omega\oint(1-\frac{E}{m\omega^2r^2}+\cdots)\,r\,dr + \cdots \\ &= -\ell\,h - 2\pi i^2 m\omega\frac{E}{m\omega^2} \end{aligned} \tag{1}$$

from which we obtain

$$E = h\omega(n+\ell)$$

An interesting side note is that the Schrödinger equation gives the same result plus an additive constant, and therefore the BSW quantization theory is in exact agreement with Schrödinger wave mechanics as far as energy differences are concerned.

A note on this procedure exerpted from the Physics 307 handout "IntegralTransform".
For the Kepler Hamiltonian

$$E = \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}p_\theta^2 - \frac{k}{r}$$

the orbit covers $0 \le \theta \le 2\pi$ and in the r variable the particle travels from the radius of closest approach (perihelion) r_p to the point of most distant departure r_a (aphelion), twice. These lead to

$$2\int_{r_p}^{r_a}\sqrt{2m\Big(E-\frac{1}{2mr^2}p_\theta^2+\frac{k}{r}\Big)}\,dr = nh, \qquad 2\pi p_\theta = \ell h$$

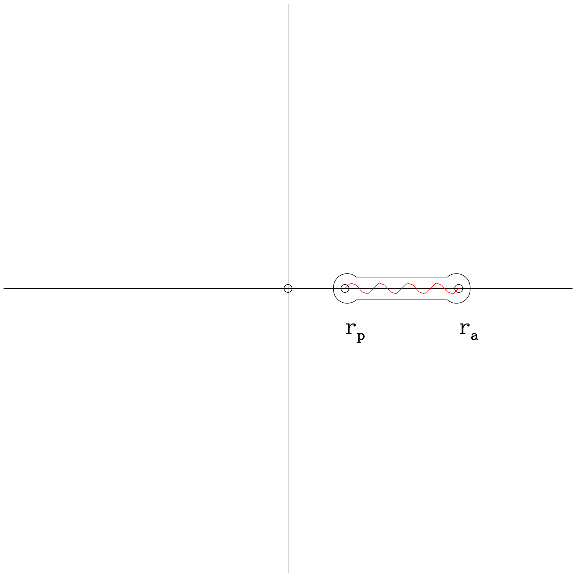
in which r_a and r_p are the inverse-roots of the radical

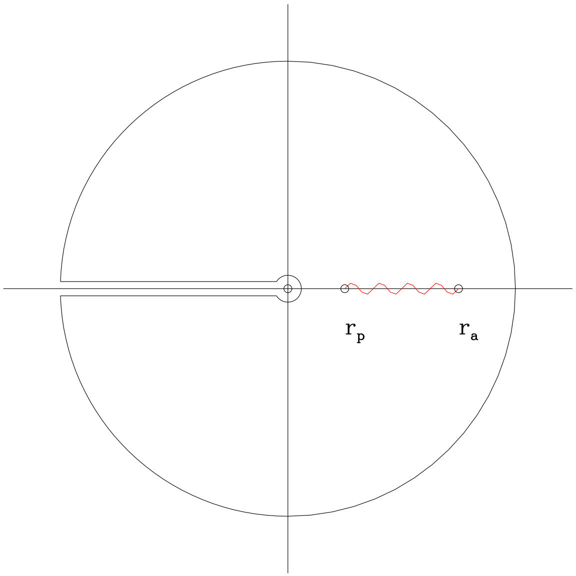
$$E - \frac{1}{2mr^2}p_\theta^2 + \frac{k}{r} = 0 \quad \text{at} \quad r_a,\, r_p$$

This integral can be computed as

$$\oint_C \sqrt{2m\Big(E-\frac{1}{2mz^2}p_\theta^2+\frac{k}{z}\Big)}\,dz = nh$$

in which the path C surrounds the branch cut of the integrand connecting the points $z = r_a, z = r_p$; Along the top of this contour, the radical is positive, and along the lower side of the cut it is negative. The contour is traveled **counterclockwise**, from r_p to r_a and back below the cut.





We perform the actual integration by deforming the contour without passing through any poles or cuts, until it looks like the following The two contributions to the integral along $z = r$, $-\infty \leq r < 0$ cancel, and we get

$$- \lim_{R\to\infty} \int_0^{2\pi} \sqrt{2m\Big(E - \frac{p_\theta^2}{2mR^2e^{2i\theta}} + \frac{k}{Re^{i\theta}}\Big)}\, iRe^{i\theta}\, d\theta + \lim_{R\to 0} \int_0^{2\pi} \sqrt{2m\Big(E - \frac{p_\theta^2}{2mR^2e^{2i\theta}} + \frac{k}{Re^{i\theta}}\Big)}\, iRe^{i\theta}\, d\theta$$

$$= nh$$

the first integral is positive since we travel clockwise around the large loop, but counterclockwise around the small. Performing appropriate expansions for each integral (keeping factors with exactly one occurance of $e^{i\theta}$ in the denominator)

$$- \lim_{R\to\infty} \int_0^{2\pi} \sqrt{2mE}\Big(1 - \frac{1}{2}\frac{p_\theta^2}{2mER^2e^{2i\theta}} + \frac{k}{2ERe^{i\theta}} + \cdots\Big)\, iRe^{i\theta}\, d\theta + \lim_{R\to 0} \int_0^{2\pi} \sqrt{-\frac{p_\theta^2}{R^2e^{2i\theta}}}\, iRe^{i\theta}\, d\theta = nh$$

and we arrive at

$$-2\pi\sqrt{2mE}\,\frac{ik}{2E} - 2\pi\ell\hbar = nh \quad \text{or} \quad E = -\frac{mk^2}{2\hbar^2(n+\ell)^2}$$

which is the celebrated Bohr-Sommerfeld-Wilson energy level formula for an electron orbiting a Hydrogen nucleus, with $k = \frac{Ze^2}{4\pi\epsilon_0}$. **This will make more sense to you if you have taken complex analysis**.

Hamilton-Jacobi theory provides us with an answer to the question “What is the Action?”. Begin with the action, which for a system of constant energy reduces to

$$S = \int_0^t \Big(p\dot{q} - H\Big)\, dt = \int_0^t \Big(p\dot{q} - E\Big)\, dt = \int_{q(0)}^{q(t)} p\, dq - Et$$

but for the Hamilton-Jacobi generating function (a modified type-2) $F(q, E, t) = F_2'(q, P, t) = W(q, E) - Et$ we know that $P = E$ and $p = \left(\frac{\partial F}{\partial q}\right)_E = \left(\frac{\partial W}{\partial q}\right)_E$, so

$$S = \int_{q(0)}^{q(t)} \Big(\frac{\partial W}{\partial q}\Big)_E\, dq - Et = W(q(t), E) - Et = F$$

Amazing, the Hamilton-Jacobi generating function for transformation into new variables $(q,p) \to (Q,P) = (\beta, E)$ is the Action S!

50. This one is done as an example in Ch. 7, so instead lets solve $H = \frac{p^2}{2m} + mgq$ subject to $q_0 = h$ and $p_0 \neq 0$;

$$\{q,H\}_{pb} = \frac{p}{m}, \quad \{\{q,H\}_{pb},H\}_{pb} = \{\frac{p}{m},H\}_{pb} = -g, \quad \{\{\{q,H\}_{pb},H\}_{pb},H\}_{pb} = 0$$

so

$$q(t) = q(0) + t\frac{p_0}{m} + \tfrac{1}{2}t^2(-g) + 0 = h + \frac{p_0 t}{m} - \tfrac{1}{2}gt^2$$

Surprise!

51. You will need the derivation rule for Poisson brackets, and $\{x,y\}_{pb} = \{x,z\}_{pb} = \{y,z\}_{pb} = 0$,$\{p_x,p_y\}_{pb} = \{p_x,p_z\}_{pb} = \{p_y,p_z\}_{pb} = 0$, $\{x_i,p_j\} = \delta_{ij}$

$$\{L_x,L_y\}_{pb} = \{yp_z - zp_y, zp_x - xp_z\}_{pb}$$

$$= \{yp_z, zp_x\}_{pb} + \{-zp_y, -xp_z\}_{pb} + \{yp_z, -xp_z\}_{pb} + \{-zp_y, zp_x\}_{pb}$$

$$= yp_x\{p_z,z\}_{pb} + p_y x\,\{z,p_z\}_{pb} + 0 + 0 = x\,p_y - y\,p_x = L_z$$

and similarly $\{L_y,L_z\}_{pb} = L_x$, $\{L_z,L_x\}_{pb} = L_y$.

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mr^2\sin^2\theta\dot\phi, \qquad p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta$$

which satisfy $\{\theta,p_\theta\}_{pb} = 1 = \{\phi,p_\phi\}_{pb}$ which you definitively proved in problem 42 (these are Q and P for a generating function given in problem 37), and of course $\{\theta,\phi\}_{pb} = \{p_\theta,p_\phi\}_{pb} = 0$ **by definition**

$$\{\theta,\phi\}_{pb} = \Big(\frac{\partial\theta}{\partial r}\frac{\partial\phi}{\partial p_r} - \frac{\partial\theta}{\partial p_r}\frac{\partial\phi}{\partial r}\Big) + \Big(\frac{\partial\theta}{\partial\theta}\frac{\partial\phi}{\partial p_\theta} - \frac{\partial\theta}{\partial p_\theta}\frac{\partial\phi}{\partial\theta}\Big) + \Big(\frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial p_\phi} - \frac{\partial\theta}{\partial p_\phi}\frac{\partial\phi}{\partial\phi}\Big) = 0$$

$$\{p_\theta,p_\phi\}_{pb} = \Big(\frac{\partial p_\theta}{\partial r}\frac{\partial p_\phi}{\partial p_r} - \frac{\partial p_\theta}{\partial p_r}\frac{\partial p_\phi}{\partial r}\Big) + \Big(\frac{\partial p_\theta}{\partial\theta}\frac{\partial p_\phi}{\partial p_\theta} - \frac{\partial p_\theta}{\partial p_\theta}\frac{\partial p_\phi}{\partial\theta}\Big) + \Big(\frac{\partial p_\theta}{\partial\phi}\frac{\partial p_\phi}{\partial p_\phi} - \frac{\partial p_\theta}{\partial p_\phi}\frac{\partial p_\phi}{\partial\phi}\Big) = 0$$

$$\{p_\theta,\theta\}_{pb} = \Big(\frac{\partial p_\theta}{\partial r}\frac{\partial\theta}{\partial p_r} - \frac{\partial p_\theta}{\partial p_r}\frac{\partial\theta}{\partial r}\Big) + \Big(\frac{\partial p_\theta}{\partial\theta}\frac{\partial\theta}{\partial p_\theta} - \frac{\partial p_\theta}{\partial p_\theta}\frac{\partial\theta}{\partial\theta}\Big) + \Big(\frac{\partial p_\theta}{\partial\phi}\frac{\partial\theta}{\partial p_\phi} - \frac{\partial p_\theta}{\partial p_\phi}\frac{\partial\theta}{\partial\phi}\Big) = -1$$

52.

$$\dot L_x = \{L_x,H\}_{pb} = \frac{1}{I_{yy}}\,L_y\,\{L_x,L_y\}_{pb} + \frac{1}{I_{zz}}\,L_z\,\{L_x,L_z\}_{pb} = \Big(\frac{1}{I_{yy}} - \frac{1}{I_{zz}}\Big)\,L_yL_z$$

Similarly

$$\dot L_y = \Big(\frac{1}{I_{zz}} - \frac{1}{I_{xx}}\Big)\,L_xL_z, \qquad \dot L_z = \Big(\frac{1}{I_{xx}} - \frac{1}{I_{yy}}\Big)\,L_xL_y$$

If $I_{xx} = I_{yy}$, we have $L_z = C$ constant, and calling $\Big(\frac{1}{I_{zz}} - \frac{1}{I_{xx}}\Big) = A$,

$$\dot L_x = -AC\,L_y, \qquad \dot L_y = AC\,L_x, \qquad L_x\dot L_x + L_y\dot L_y = 0, \qquad L_x^2 + L_y^2 = b$$

which is an oscillator

$$\frac{d}{dt}(L_x + iL_y) = iAC(L_x + iL_y), \qquad L_x + iL_y = C_+\,e^{iAC\,t}$$

$$\frac{d}{dt}(L_x - iL_y) = iAC(-L_x + iL_y), \qquad L_x - iL_y = C_-\, e^{-iAC\,t}$$

$$L_x = \tfrac{1}{2}\Big(C_+\, e^{iAC\,t} + C_-\, e^{-iAC\,t}\Big)$$

but its not your garden-variety oscillator; note that the "frequency" $\omega = AC$ depends on $C = L_z$, the angular momentum components L_x and L_y precess around L_z, precession is fast for long L_z components.

53. Calculate using the derivation rule for Poisson brackets (you can ignore things that evaluate to zero like $\{Q, Q^2\}_{pb}$)

$$\begin{aligned}
\{L_+, L_-\}_{pb} &= 2S\{P,Q\}_{pb} - Q\{P^2, Q\}_{pb} = 2S\{P,Q\}_{pb} - 2QP\{P,Q\}_{pb} \\
&= -2S + 2PQ = 2(PQ - S) = 2L_3 \\
\{L_3, L_-\}_{pb} &= \{P,Q\}_{pb}Q = -Q = -L_- \\
\{L_3, L_+\}_{pb} &= 2S\{PQ, P\}_{pb} - \{PQ, PPQ\}_{pb} = 2SP\{Q,P\}_{pb} - P\{PQ,PQ\}_{pb} - PQ\{PQ,P\}_{pb} \\
&= 2SP - 0 - P^2Q\{Q,P\}_{pb} = 2SP - P^2Q = L_+ \qquad (2)
\end{aligned}$$

This is an interesting problem, note that this is an angular momentum Lie-Poisson algebra

$$\{\tfrac{1}{2}(L_+ + L_-), L_3\}_{pb} = \tfrac{1}{2}(L_+ - L_-), \qquad \{\tfrac{1}{2}(L_+ - L_-), L_3\}_{pb} = \tfrac{1}{2}(L_+ + L_-)$$

$$\{\tfrac{1}{2}(L_+ + L_-), \tfrac{1}{2}(L_+ - L_-)\}_{pb} = \tfrac{1}{4}\Big(- \{L_+, L_-\}_{pb} + \{L_- L_+\}_{pb}\Big) = -\tfrac{1}{2}\{L_+, L_-\}_{pb} = -L_3$$

and so comparing with problem 51, we have an isomorphism

$$L_3 = L_z, \qquad \tfrac{1}{2}(L_+ + L_-) = L_y, \qquad \tfrac{1}{2}(L_+ - L_-) = L_x$$

In other words we have used a single set of canonically conjugate variables (Q, P) to construct a set of three non-canonical angular momentum variables!

54.

$$\begin{aligned}
\{L_+, L_-\}_{pb} &= \{P_1Q_2, P_2Q_1\}_{pb} = P_1\{Q_2, P_2Q_1\}_{pb} + Q_2\{P_1, P_2Q_1\}_{pb} \\
&= P_1Q_1\{Q_2, P_2\}_{pb} + Q_2P_2\{P_1, Q_1\}_{pb} = P_1Q_1 - P_2Q_2 = 2L_3 \\
\{L_3, L_+\}_{pb} &= \tfrac{1}{2}\{P_1Q_1, P_1Q_2\}_{pb} - \tfrac{1}{2}\{P_2Q_2, P_1Q_2\}_{pb} \\
&= \tfrac{1}{2}P_1Q_2\{Q_1, P_1\}_{pb} - \tfrac{1}{2}Q_2P_1\{P_2, Q_2\}_{pb} = P_1Q_2 = L_+ \\
\{L_3, L_-\}_{pb} &= \tfrac{1}{2}\{P_1Q_1, P_2Q_1\}_{pb} - \tfrac{1}{2}\{P_2Q_2, P_2Q_1\}_{pb} \\
&= \tfrac{1}{2}P_2Q_1\{P_1, Q_1\}_{pb} - \tfrac{1}{2}Q_1P_2\{Q_2, P_2\}_{pb} = -P_2Q_1 = L_- \qquad (3)
\end{aligned}$$

The techniques used to compute Poisson brackets are almost identical to those used to compute Lie brackets or commutators **except** that with Lie brackets, order is important. The order in which classical variables are multiplied together is not important, since $PQ = QP$, but order is very important for commutators, since for **operators** $\hat{p}\hat{q} \neq \hat{q}\hat{p}$.
You should (for future reference) realize that the derivation rule for Poisson brackets can be "iterated" to

$$\{Q, P\}_{pb} = 1, \quad \{Q, P^2\}_{pb} = \{Q, P\}_{pb}P + \{Q, P\}_{pb}P = 2P, \qquad \{Q, P^n\}_{pb} = nP^{n-1}$$

and therefore, by expanding any $f(Q,P)$ in a series in P; $f(Q,P)=\sum_n a_n(Q)\,P^n$

$$\{Q, f(Q,P)\_pb = \sum_n a_n(Q)\,\{Q,P^n\}_{pb} = \sum_n a_n(Q)\,n\,P^{n-1} = \frac{\partial f}{\partial P}$$

In other words, the FPR $\{Q,P\}_{pb}=1$ together with the derivation rule is responsible for the derivative form of the Poisson bracket

$$\{f(Q,P),g(Q,P)\}_{PB} = \Big(\frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\Big)\,\{Q,P\}_{pb}$$

so that the FPR and derivation rule **alone** are enough to define all Poisson brackets.

55. Apply these to a function (see Eq. 4.23 if you want to save a lot of work, or Eq. 7.29)

$$\begin{aligned}
[\mathbf{v}_1,\mathbf{v}_2]f &= \mathbf{v_1}(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}) - \mathbf{v}_2(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}) \\
&= (y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}) - (z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}) \\
&= \Big(y(\frac{\partial f}{\partial x} + z\frac{\partial^2 f}{\partial x\partial z}) - xy\frac{\partial^2 f}{\partial z^2} - z^2\frac{\partial^2 f}{\partial x\partial y} + zx\frac{\partial^2 f}{\partial y\partial z}\Big) \\
&- \Big(x(\frac{\partial f}{\partial y} + z\frac{\partial^2 f}{\partial y\partial z}) - xy\frac{\partial^2 f}{\partial z^2} - z^2\frac{\partial^2 f}{\partial x\partial y} + zy\frac{\partial^2 f}{\partial x\partial z}\Big) \\
&= y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x} = -\mathbf{v}_3 f
\end{aligned} \tag{4}$$

similarly

$$[\mathbf{v}_2,\mathbf{v}_3] = -\mathbf{v}_1, \qquad [\mathbf{v}_3,\mathbf{v}_1] = -\mathbf{v}_2$$

so that (up to a sign) these tangent vectors form an angular-momentum Lie algebra. You will learn that up to a factor of $i\hbar$ these are the quantum angular momentum **operators**.
56. The proof is equation 4.23.
57.

58. The Hamiltonian is $H=-\mu_B B_3\,x_3$

$$\dot{x}_1 = \{x_1,H\}_{pb} = -\mu_B B_3\,\{x_1,x_3\}_{pb} = \mu_B B_3\,x_2, \qquad \dot{x}_2 = \{x_2,H\}_{pb} = -\mu_B B_3\,\{x_2,x_3\}_{pb} = -\mu_B B_3\,x_1, \qquad \dot{x}_3 = 0$$

which we can re-write as

$$\dot{\mathbf{x}} = (\dot{x}_1,\dot{x}_2,\dot{x}_3) = \mu_B B_3\,\mathbf{n}\times\mathbf{x}, \qquad \mathbf{n}=(0,0,-1)$$

Now use problem 18

$$\xi = x_1 + ix_2, \quad \dot{\xi} = \dot{x}_1 + i\dot{x}_2 = \mu_B B_3(x_2 - ix_1) = -i\mu_B B_3(x_1+ix_2) = -i\mu_B B_3\,\xi$$

so $\xi(t) = (x_1(t)+ix_2(t)) = \xi(0)\,e^{-i\mu_B B_3\,t}$ and of course $\xi^*(t) = (x_1(t)-ix_2(t)) = \xi^*(0)\,e^{i\mu_B B_3\,t}$, therefore

$$x_1(t) = \tfrac{1}{2}\Big(\xi(0)\,e^{-i\mu_B B_3\,t} + \xi^*(0)\,e^{i\mu_B B_3\,t}\Big), \qquad x_2(t) = \frac{1}{2i}\Big(\xi(0)\,e^{-i\mu_B B_3\,t} - \xi^*(0)\,e^{i\mu_B B_3\,t}\Big), \qquad x_3(t) = x_3(0)$$

What is the energy? It is given by H; $E=H=-\mu_B B_3\,x_3(t) = -\mu_B B_3\,x_3(0)$, which is clearly constant (conserved). You can see that a precessing system reduces dynamically to a harmonic oscillator.

59. Equations of motion are

$$\dot{x}_1 = J(y_2x_3 - y_3x_2), \quad \dot{x}_2 = J(y_3x_1 - y_1x_3), \quad \dot{x}_3 = J(y_1x_2 - y_2x_1)$$

$$\dot{y}_1 = -J(y_2x_3 - y_3x_2), \quad \dot{y}_2 = -J(y_3x_1 - y_1x_3), \quad \dot{y}_3 = -J(y_1x_2 - y_2x_1)$$

which can be written as

$$\dot{\mathbf{x}} = -J\,\mathbf{y} \times \mathbf{x}, \qquad \dot{\mathbf{y}} = J\,\mathbf{y} \times \mathbf{x}$$

You immediately get three conserved quantities

$$\dot{\mathbf{x}} + \dot{\mathbf{y}} = 0, \qquad x_1 + y_1 = S_1, \quad x_2 + y_2 = S_2, \quad x_3 + y_3 = S_3$$

Put these back into the EOMs for x_i;

$$\dot{\mathbf{x}} = -J\Big(\mathbf{S} - \mathbf{x}\Big) \times \mathbf{x} = -J\mathbf{S} \times \mathbf{x}$$

which is a precession equation; $\mathbf{x}$ precesses around $\frac{\mathbf{S}}{|\mathbf{S}|}$ at rate $\Omega = J|\mathbf{S}|$. Computing the energy is easy;

$$H = E = -J\mathbf{x} \cdot \mathbf{y}, \qquad |\mathbf{S}|^2 = \Big(\mathbf{x} + \mathbf{y}\Big) \cdot \Big(\mathbf{x} + \mathbf{y}\Big) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x} \cdot \mathbf{y} = 2 - 2\frac{H}{J}$$

so that

$$H = J\Big(1 - \tfrac{1}{2}|\mathbf{S}|^2\Big)$$

The fact that $|\mathbf{S}| \leq 2$ follows from the Schwartz inequality.

60. Use the derivation rule for Poisson brackets, the equations of motion are

$$\dot{x}_k = J\sum_{i,j}\Big(\{x_k, y_i\}_{pb}y_j + \{x_k, y_j\}_{pb}y_i + \{x_k, z_i\}_{pb}z_j + \{x_k, z_j\}_{pb}z_i\Big)$$

$$= J\sum_{i,j}\Big(\delta_{ki}\, z_k y_j + \delta_{kj}\, z_k y_i - \delta_{ki}\, y_k z_j - \delta_{kj}\, y_k z_i\Big)$$

$$= 2Jz_k\sum_j y_j - 2Jy_k\sum_j z_j$$

and so forth. Call $\sum_j(x_j, y_j, z_j) = \mathbf{S}$, and $(x_j, y_j, z_j) = \mathbf{r}_j$, then

$$\dot{x}_k = -2J\Big(\mathbf{S}_z y_k - \mathbf{S}_y z_k\Big) = -2J\Big(\mathbf{S} \times \mathbf{r}\Big)_k$$

which says that each $\mathbf{r}_j$ precesses at the same rate around $\mathbf{S}$, and $\mathbf{S} = \sum_j \mathbf{r}_j$ is conserved

$$\dot{\mathbf{S}} = -2J\mathbf{S} \times \mathbf{S} = 0$$

It is as if all of the spins combine to form a magnetic field about which every spin precesses, which is why it is called a mean-field model.

61. This is a simple problem, but it is very cool. It is trivial to do in the case of $H = -Bx_3$;

$$\dot{x}_1 = \{x_1, -Bx_3\}_{pb} = Bx_2, \quad \dot{x}_2 = \{x_2, -Bx_3\}_{pb} = -Bx_1, \quad \dot{x}_3 = \{x_3, -Bx_3\}_{pb} = 0$$

because the equations of motion are already linear, we get

$$\dot{x}_1 + i\dot{x}_2 \approx 2\dot{\pi} = -iB(x_1 + ix_2) \approx -iB\pi, \qquad \pi(t) = \pi(0)e^{-iBt}$$

Lets instead linearize the equations of motion of problem 52;

$$\dot{x}_1 = \Big(\frac{1}{I_{yy}} - \frac{1}{I_{zz}}\Big)x_2x_3 = Ax_2x_3, \qquad \dot{x}_2 = \Big(\frac{1}{I_{zz}} - \frac{1}{I_{xx}}\Big)x_1x_3 = Bx_1x_3, \qquad \dot{x}_3 = \Big(\frac{1}{I_{xx}} - \frac{1}{I_{yy}}\Big)x_1x_2 = (A-B)x_1x_2$$

These become (in the Darboux frame with $x_3 \approx -1$ and x_1, x_2 small)

$$\dot{x}_1 \approx -Ax_2, \qquad \dot{x}_2 = -B\,x_1, \qquad \dot{x}_3 \approx 0$$

Why are these interesting? One or the other but not both of A and B must be negative, or the system does not have oscillatory solutions, and x_1 and x_2 will not remain small, they will grow and migrate away from the neighborhood of $(0, 0, -1)$.

62. Simply calculate;

$$d\boldsymbol{\omega} = \frac{\partial A}{\partial x_3}dx_3 \wedge dx_1 \wedge dx_2 + \frac{\partial B}{\partial x_1}dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial C}{\partial x_2}dx_2 \wedge dx_3 \wedge dx_1 = \Big(\frac{\partial A}{\partial x_3} + \frac{\partial B}{\partial x_1} + \frac{\partial C}{\partial x_2}\Big)\, dx_1 \wedge dx_2 \wedge dx_3$$

which really does not look like zero, **but you can't have a three-form on a two-dimensional manifold**;

$$d(x_1^2 + x_2^2 + x_3^2) = d(1) = 0, \quad x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3 = 0, \quad \text{so} \quad dx_3 = -\frac{1}{x_3}\Big(x_1\,dx_1 + x_2\,dx_2\Big)$$

therefore (since $dx^1 \wedge dx^1 = dx^2 \wedge dx^2 = 0$)

$$d\boldsymbol{\omega} = -\frac{1}{x_3}\Big(\frac{\partial A}{\partial x_3} + \frac{\partial B}{\partial x_1} + \frac{\partial C}{\partial x_2}\Big)\, dx_1 \wedge dx_2 \wedge \Big(x_1\,dx_1 + x_2\,dx_2\Big) = 0$$

63.

$$\dot{\theta}_1 = \tfrac{1}{2}\Big(B_2\theta_3 - B_3\theta_2\Big), \quad \dot{\theta}_2 = -\tfrac{1}{2}\Big(B_1\theta_3 - B_3\theta_1\Big), \quad \dot{\theta}_2 = \tfrac{1}{2}\Big(B_1\theta_2 - B_2\theta_1\Big)$$

which does indeed describe a vector $(\theta_1, \theta_2, \theta_3)$ precessing about (B_1, B_2, B_3). For the next part, just compute time derivatives

$$\dot{s}_3 = \dot{\theta}_1\theta_2 + \theta_1\dot{\theta}_2 = \tfrac{1}{2}\Big(B_2\theta_3\,\theta_2 - B_1\theta_1\,\theta_3\Big) = \tfrac{1}{2}(B_1\,s_2 - B_2\,s_1\Big)$$

and so forth, another set of precessing objects. Compute the Poisson brackets from the formula;

$$\{s_1, s_2\}_{pb} = 2\sum_i \Big((\theta_2\theta_3)\Big(\overleftarrow{\frac{\partial}{\partial\theta_i}}\Big)\Big)\Big(\Big(\overrightarrow{\frac{\partial}{\partial\theta_i}}\Big)(\theta_3\theta_1)\Big)$$

$$= 2\sum_i \Big(\theta_2\,\delta_{i3} - \theta_3\,\delta_{i2}\Big)\Big(\theta_1\,\delta_{i3} - \theta_3\,\delta_{i1}\Big) = 2\,\theta_2\,\theta_1 = -2\,s_3$$

64. Equations of motion look like directional derivatives of H, which of course they are

$$\dot{x}^k = \omega^{ki}\frac{\partial H}{\partial x^i}$$

65.

$$d\boldsymbol{\omega} = \sum_{ij} d\omega_{ij}(x) \wedge dx^i \wedge dx^j = \sum_{ijk} \frac{\partial \omega_{ij}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j = 0$$

For example, suppose that we have three x's;

$$\boldsymbol{\omega} = \Big(\omega_{12} - \omega_{21}\Big)\, dx^1 \wedge dx^2 + \Big(\omega_{13} - \omega_{31}\Big)\, dx^1 \wedge dx^3 + \Big(\omega_{23} - \omega_{32}\Big)\, dx^2 \wedge dx^3$$

It must be that $\omega_{ij} = -\omega_{ji}$, because any part of ω_{ij} that would be symmetric as opposed to antisymmetric would drop out of this formula, so

$$\boldsymbol{\omega} = 2\Big(\omega_{12}\, dx^1 \wedge dx^2 + \omega_{13}\, dx^1 \wedge dx^3 + \omega_{23}\, dx^2 \wedge dx^3\Big)$$

$$d\boldsymbol{\omega} = 2\Big(\frac{\partial \omega_{12}}{\partial x^3}\, dx^3 \wedge dx^1 \wedge dx^2 + \frac{\partial \omega_{13}}{\partial x^2}\, dx^2 \wedge dx^1 \wedge dx^3 + \frac{\partial \omega_{23}}{\partial x^1}\, dx^1 \wedge dx^2 \wedge dx^3\Big)$$

$$= 2\Big(\frac{\partial \omega_{12}}{\partial x^3} + \frac{\partial \omega_{31}}{\partial x^2} + \frac{\partial \omega_{23}}{\partial x^1}\Big)\, dx^1 \wedge dx^2 \wedge dx^3 = 0, \qquad \frac{\partial \omega_{12}}{\partial x^3} + \frac{\partial \omega_{31}}{\partial x^2} + \frac{\partial \omega_{23}}{\partial x^1} = 0$$

(note the second term).

66. Just compute Poisson brackets

$$\begin{aligned}
\dot{x}_j &= \{x_j, -J\sum_i (x_i, x_{i+1} + y_i y_{i+1} + z_i z_{i+1}\}_{pb} \\
&= -J\sum_i \Big(\{x_j, y_i\}_{pb} y_{i+1} + \{x_j, y_{i+1}\}_{pb} y_i + \{x_j, z_i\}_{pb} z_{i+1} + \{x_j, z_{i+1}\}_{pb} z_i\Big) \\
&= -J\sum_i \Big(\delta_{ij} z_j y_{i+1} + \delta_{j,i+1} z_j y_i - \delta_{ij} y_j z_{i+1} - \delta_{j,i+1} y_j z_i\Big) \\
&= -J\Big(z_j y_{j+1} + z_j y_{j-1} - y_j z_{j+1} - y_j z_{j-1}\Big) \\
\dot{y}_j &= -J\Big(x_j z_{j+1} + x_j z_{j-1} - z_j x_{j+1} - z_j x_{j-1}\Big) \\
\dot{z}_j &= -J\Big(y_j x_{j+1} + y_j x_{j-1} - x_j y_{j+1} - x_j y_{j-1}\Big) \qquad (5)
\end{aligned}$$

(the neighbors $j \pm 1$ provide the field about which spin-j precesses). Now pass to a Darboux frame near $(x_j, y_j, z_j) = (0, 0, S)$

$$x_j + i y_j \approx \sqrt{2S}\, \chi_j, \qquad x_j - i y_j \approx \sqrt{2S}\, \pi_j, \qquad z_j \approx S$$

$$\begin{aligned}
\dot{x}_j &\approx -JS\Big(y_{j+1} + y_{j-1} - y_j - y_j\Big) \\
\dot{y}_j &\approx -JS\Big(x_j + x_j - x_{j+1} - x_{j-1}\Big) \\
\dot{x}_j + i\dot{y}_j = \sqrt{2S}\, \dot{\chi}_j &\approx -(-i) JS\sqrt{2S}\Big(\chi_{j+1} + \chi_{j-1} - 2\chi_j\Big) \\
\dot{x}_j - i\dot{y}_j = \sqrt{2S}\, \dot{\pi}_j &\approx -(i) JS\sqrt{2S}\Big(\pi_{j+1} + \pi_{j-1} - 2\pi_j\Big) \\
\dot{z}_j &\approx 0 \qquad (6)
\end{aligned}$$

(because both x and y are small, the product is negligible). Nearest-neighbor interactions **can always be "diagonalized" by Fourier transformation**; let

$$\chi_j(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{i(\frac{2\pi k}{N} j - \Omega(k) t)}\, \eta_k$$

$$\frac{1}{\sqrt{N}}\sum_{k=1}^{N} -i\Omega(k)\, e^{i(\frac{2\pi k}{N}j-\Omega(k)t)}\,\eta_k = iJS\sum_{k=1}^{N}\Big(e^{i(\frac{2\pi k}{N}(j+1)-\Omega(k)t)}\,\eta_k + e^{i(\frac{2\pi k}{N}(j-1)-\Omega(k)t)}\,\eta_k - 2e^{i(\frac{2\pi k}{N}j-\Omega(k)t)}\,\eta_k\Big)$$

$$= iJS\sum_{k=1}^{N} e^{i(\frac{2\pi k}{N}j-\Omega(k)t)}\,\eta_k\Big(e^{i\frac{2\pi k}{N}} + e^{-i\frac{2\pi k}{N}} - 2\Big)$$

from which we can see that the coefficients of different independent oscillatory functions are

$$\Omega(k) = JS\Big(2 - e^{i\frac{2\pi k}{N}} - e^{-i\frac{2\pi k}{N}}\Big) = 2JS\Big(1-\cos(\frac{2\pi k}{N})\Big)$$

this is the spectrum of “small amplitude” spin-waves or magnons.